ENGLISH SCHOOL CLASSICS

EDITED BY FRANCIS STORR, B.A.,

CHIEF MASTER OF MODERN SUBJECTS IN MERCHANT TAYLORS' SCHOOL.

THOMSON'S SEASONS: Winter.
With an Introduction to the Series. By the Rev. J. F. BRIGHT, M.A. 1s.

COWPER'S TASK.
By FRANCIS STORR, B.A. 2s.; or in Three Parts, 9d. each.

COWPER'S SIMPLE POEMS.
By FRANCIS STORR, B.A. 1s.

SCOTT'S LAY OF THE LAST MINSTREL.
By J. SURTEES PHILLPOTTS, M.A., Head Master of Bedford School. 2s. 6d.; or in Four Parts, 9d. each.

SCOTT'S LADY OF THE LAKE.
By R. W. TAYLOR, M.A., Head Master of Kelly College, Tavistock. 2s. or in Three Parts, 9d. each.

SCOTT'S MARMION.
By F. S. ARNOLD, M.A., Assistant Master at Bedford School.

NOTES TO SCOTT'S WAVERLEY.
By H. W. EVE, M.A., Head Master of University College School, London. 1s.; WAVERLEY AND NOTES, 2s. 6d.

BACON'S ESSAYS. Complete Edition.
By FRANCIS STORR, B.A.

TWENTY OF BACON'S ESSAYS.
By FRANCIS STORR, B.A. 1s.

SIMPLE POEMS.
By W. E. MULLINS, M.A., Assistant Master at Marlborough College. 8d.

SELECTIONS FROM WORDSWORTH'S POEMS.
By H. H. TURNER, B.A., late Scholar of Trinity College, Cambridge. 1s.

WORDSWORTH'S EXCURSION: The Wanderer.
By H. H. TURNER, B.A. 1s.

MILTON'S PARADISE LOST.
By FRANCIS STORR, B.A. Book I., 9d. Book II., 9d.

MILTON'S L'ALLEGRO, IL PENSEROSO, AND LYCIDAS.
By EDWARD STORR, M.A., late Scholar of New College, Oxford. 1s.

SELECTIONS FROM THE SPECTATOR.
By OSMUND AIRY, M.A., late Assistant Master at Wellington College. 1s.

BROWNE'S RELIGIO MEDICI.
By W. P. SMITH, M.A., Assistant Master at Winchester College. 1s.

GOLDSMITH'S TRAVELLER AND DESERTED VILLAGE.
By C. SANKEY, M.A., Head Master of Bury St. Edmund's Grammar School. 1s.

EXTRACTS from GOLDSMITH'S VICAR OF WAKEFIELD.
By C. SANKEY, M.A. 1s.

POEMS SELECTED from the WORKS OF ROBERT BURNS.
By A. M. BELL, M.A., Balliol College, Oxford. 2s.

MACAULAY'S ESSAYS:
MOORE'S LIFE OF BYRON. By FRANCIS STORR, B.A. 9d.
BOSWELL'S LIFE OF JOHNSON. By FRANCIS STORR, B.A. 9d.
HALLAM'S CONSTITUTIONAL HISTORY. By H. F. BOYD, late Scholar of Brasenose College, Oxford. 1s.

SOUTHEY'S LIFE OF NELSON.
By W. E. MULLINS, M.A. 2s. 6d.

GRAY'S POEMS. SELECTION FROM LETTERS, with LIFE by JOHNSON. By FRANCIS STORR, B.A. 1s.

Waterloo Place, Pall Mall, London.

Rivingtons' Mathematical Series

KEY TO ELEMENTS OF GEOMETRY

Rivington's Mathematical Series.

By J. HAMBLIN SMITH, M.A.,

of Gonville and Caius College, and late Lecturer of St. Peter's College, Cambridge.

Small 8vo.

A Treatise on Arithmetic. 3*s.* 6*d.*
(Copies may be had without the Answers.)

A Key to Arithmetic. *Crown* 8*vo.* 9*s.*

Elementary Algebra. *Part I.* 3*s.*
Without Answers, 2*s.* 6*d.*

A Key to Elementary Algebra. *Crown* 8*vo.* 9*s.*

Exercises on Algebra. 2*s.* 6*d.*
(Copies may be had without the Answers.)

Elements of Geometry. *Containing Books* 1 *to* 6, *and portions of Books* 11 *and* 12 *of Euclid, with Exercises and Notes, arranged with the Abbreviations admitted in the Cambridge Examinations.* 3*s.* 6*d.*

PART I. Containing Books 1 *and* 2 *of Euclid, limp cloth,* 1*s.* 6*d.*, *may be had separately.*

A Key to Elements of Geometry. *Crown* 8*vo.* 8*s.* 6*d.*

Elementary Statics. 3*s.*

Elementary Hydrostatics. 3*s.*

A Key to Statics and Hydrostatics. *Cr.* 8*vo.* 6*s.*

Elementary Trigonometry. 4*s.* 6*d.*

A Key to Elementary Trigonometry. *Crown* 8*vo.* 7*s.* 6*d.*

Book of Enunciations. *For Geometry, Algebra, Trigonometry, Statics, and Hydrostatics,* 1*s.*

An Introduction to the Study of Heat. 3*s.*

By E. J. GROSS, M.A.,

Fellow of Gonville and Caius College, Cambridge, and Secretary to the Oxford and Cambridge Schools Examination Board.

Algebra. *Part II. Crown* 8*vo.* 8*s.* 6*d.*

Kinematics and Kinetics. *Crown* 8*vo.* 5*s.* 6*d.*

By G. RICHARDSON, M.A.,

Assistant Master at Winchester College, and late Fellow of St. John's College, Cambridge.

Geometrical Conic Sections. *Crown* 8*vo.* 4*s.* 6*d.*

A KEY

TO

THE EXERCISES

IN

ELEMENTS OF GEOMETRY

BY

J. HAMBLIN SMITH, M.A.

OF GONVILLE AND CAIUS COLLEGE

AND LATE LECTURER AT ST. PETER'S COLLEGE, CAMBRIDGE

New Edition

RIVINGTONS

WATERLOO PLACE, LONDON

MDCCCLXXXV

TABLE OF REFERENCE TO THE EXERCISES IN THE ELEMENTS OF GEOMETRY.

NOTE.

A SMALL number of Riders and Problems in the Edition of my *Elements of Geometry* published in 1880 will be replaced by other Exercises in future Editions. The following List points out the Exercises in the Edition of 1880 of which no solution is given in this book :—

Page.	Exercise.	Page.	Exercise.
63	2	274	5
64	2	274	8
66	1	295	19
118	33	297	40
119	34	298	48
119	39	299	59
119	44	302	84
127	Ex.	303	91
148	Ex. 1	305	105
174	47	306	115
174	49		

J. HAMBLIN SMITH.

CAMBRIDGE, *April* 1880.

KEY TO

ELEMENTARY GEOMETRY.

Page 12.

EXERCISE 1. Let F be the other point in which the circles intersect. Draw the straight lines AF, BF.

Then $\because$ A is the centre of $\odot$ BFD,

$\therefore$ $AF=AB$;

and $\because$ B is the centre of $\odot$ AFE,

$\therefore$ $BF=AB$.

$\therefore$ $AF=BF$, and ABF is an equilateral $\triangle$.

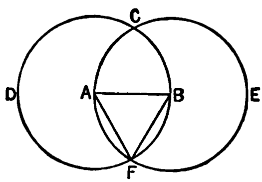


FIG. 1.

Ex. 2. Let AB, CD be two given straight lines, of which AB is the less.

Draw the straight line $AE=CD$, and with centre A and distance AE describe $\odot$ EFH.

Produce AB to meet the $\bigcirc$ce in H.

Then $AH=AE$, and $\therefore$ $AH=CD$.

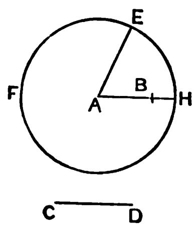


FIG. 2.

Ex. 3. With centre B and distance BC describe $\odot$ CDE, and from B draw any line BD to meet the $\bigcirc$ce in D.

Then $BD=BC$.

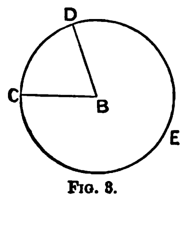


FIG. 3.

Ex. 4. Let AB be the given straight line on which the isosceles triangle is to be described, and let CD be the given straight line to which the equal sides of the triangle are to be equal.

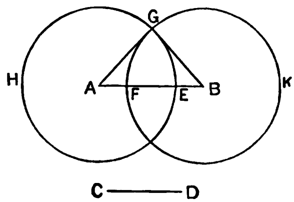


Fig. 4.

In AB, or AB produced, take $AE=CD$.

In BA, or BA produced, take $BF=CD$.

With centre A and distance AE describe the ⊙ EGH.

With centre B and distance BF describe the ⊙ FGK.

Draw the straight lines AG, BG.

Then $\because AG=AE, \therefore AG=CD$;

and $\because BG=BF, \therefore BG=CD$;

$\therefore AGB$ is a $\triangle$ described as was required.

Page 20.

Exercise 1. Taking the diagram of Prop. IX.

$$\because AE=AD, \therefore \angle ADE = \angle AED;$$

$$\text{and } \because FD=FE, \therefore \angle FED = \angle FDE;$$

$$\therefore \angle ADE \text{ with } \angle FDE = \angle AED \text{ with } \angle FED;$$

$$\therefore \angle ADF = \angle AEF.$$

$$\text{Then } \because AD=AE, \text{ and } FD=FE, \text{ and } \angle ADF = \angle AEF,$$

$$\therefore \angle EAF = \angle DAF.$$

Ex. 2. If the vertex F fall *within* the triangle DAE, or *without* the triangle DAE, the proof given in the proposition will hold good. But if the vertex F fall *on* A, the construction will fail.

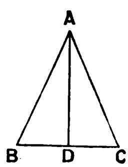


Fig. 5.

Page 21.

Exercise 1. In the $\triangle ABC$, let $AB=AC$.

Let AD bisect $\angle BAC$ and meet BC in D.

Then $\because AB=AC$, and AD is common,

and $\angle BAD = \angle CAD$;

$\therefore BD=CD$.

Ex. 2. In the diagram to Ex. 1, suppose that AD bisects BC in D.

Then $\because AB = AC$. and AD is common, and $BD = CD$,

$$\therefore \angle BAD = \angle CAD.$$

Ex. 3. Let AB be the given straight line.

Bisect AB in C, and with centre B and distance BC describe the ⊙ CDE. Produce AB to meet the ○ce in E.

Then $\because BE = CB$,

$\therefore BE$ is one-third of AE.

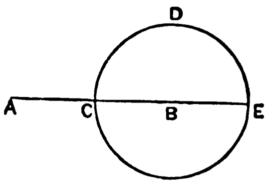


Fig. 6.

Page 22.

Exercise 1. Taking the diagram of Prop. IX., let O be the point in which AF and ED intersect.

Then $\because AD = AE$, and AO is common,

and $\angle DAO = \angle EAO$,

$\therefore DO = EO$, and $\angle AOD = \angle AOE$;

$\therefore AF$ bisects DE at right angles.

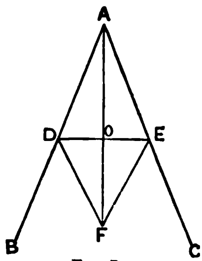


Fig. 7.

Ex. 2. Let DO, EO be two lines bisecting AB, AC, two sides of the equilateral △ ABC at right angles.

Join AO, BO, CO.

Then $\because AD = BD$, and OD is common,

and $\angle ADO = \angle BDO$,

$\therefore OA = OB$.

Similarly, it may be shown that $OA = OC$.

$\therefore OA$, OB, OC are all equal.

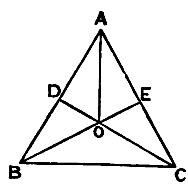


Fig. 8.

Ex. 3. In Prop. IX. we draw a straight line AF making equal angles with each of two straight lines BA, CA which meet.

In PROP. XI. we draw a straight line FC making equal angles with each of two straight lines CA, CB which meet, *and are in the same straight line.*

Page 23.

EXERCISE 1. Because the point C might be in such a position that it would be impossible to draw from it a line perpendicular to AB.

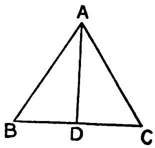


FIG. 9.

Ex. 2. In the $\triangle ABC$ let AD be a perpendicular on BC bisecting it.

Then $\because BD = CD$, and AD is common,

and $\angle ADB = \angle ADC$,

$\therefore AB = AC$.

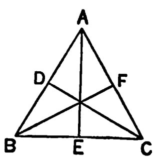


FIG. 10.

Ex. 3. Let D, E, F be the middle points of AB, BC, CA, the sides of an equilateral $\triangle$.

Then in $\triangle$s BAF, CBD,

$\because BA = CB$, and $AF = BD$,

and $\angle BAF = \angle CBD$,

$\therefore BF = CD$.

Similarly, it may be shown that $BF = AE$.

Page 24.

Ex. 1.

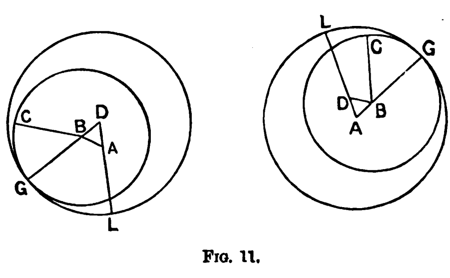


FIG. 11.

Ex. 2. Let BAC be the given angle.

Bisect $\angle BAC$ by the straight line AD.

Then bisect $\angle BAD$ by the straight line AE; and bisect $\angle CAD$ by the straight line AF.

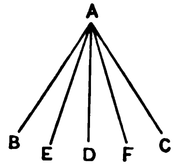


FIG. 12.

Ex. 3. Let ABC be the given isosceles triangle.

Then since $\angle DBC$ is half of $\angle ABC$,

and $\angle DCB$ is half of $\angle ACB$,

$\therefore \angle DBC = \angle DCB$, (Ax. 7.)

and $\therefore DB = DC$.

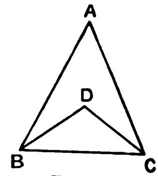


FIG. 13.

Ex. 4. In the $\triangle$s FBD, DCE,

$\because FB = DC$, and $BD = CE$,

and $\angle FBD = \angle DCE$,

$\therefore FD = ED$.

Similarly, it may be shown that $FD = FE$.

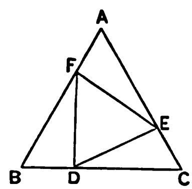


FIG. 14.

Ex. 5. Let AB be the given straight line, C and D the given points.

Join C, D by the straight line CD; and bisect CD in E.

Draw EF at right $\angle$s to CD, and let EF meet AB in F.

Join CF, DF.

Then $\because CE = DE$, and EF is common,

and $\angle CEF = \angle DEF$,

$\therefore CF = DF$.

Note.—If C and D be so situated that the line joining them *is perpendicular* to AB, the *problem is impossible.*

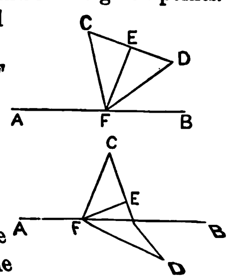


FIG. 15.

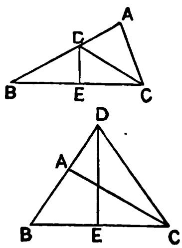


Fig. 16.

Ex. 6. Let ABC be the given $\triangle$, having the angle ABC an acute $\angle$.

Bisect BC in E, and draw $ED \perp$ to BC, meeting BA or BA produced in D. Join DC.

Then $\because BE = CE$, and ED is common,

and $\angle BED = \angle CED$,

$\therefore DB = DC$.

Ex. 7. In $\triangle$s AFC, AGB,

$\because FA = GA$, and $AC = AB$, and $\angle FAC = \angle GAB$,

$\therefore FC = GB$, and $\angle BFC = \angle CGB$.

Next, in $\triangle$s BFC, CGB,

$\because BF = CG$, and $FC = GB$, and $\angle BFC = \angle CGB$,

$\therefore \angle FBC = \angle GCB$, and $\angle BCF = \angle CBG$.

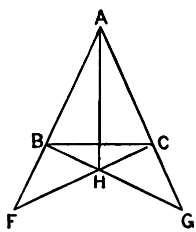


Fig. 17.

From $\angle FBC$ take $\angle CBG$, and from $\angle GCB$ take $\angle BCF$;

then $\angle FBH = \angle GCH$.

Then in $\triangle$s FBH, GCH,

$\because \angle BFH = \angle CGH$, and $\angle FBH = \angle GCH$,

and $FB = GC$,

$\therefore BH = CH$, and $FH = GH$. (I. B.)

Then in $\triangle$s AFH, AGH,

$\because AF = AG$, and AH is common, and $FH = GH$,

$\therefore \angle HAF = \angle HAG$.

Ex. 8. Let AD cut BC in E.

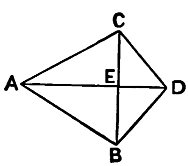


Fig. 18.

Then in $\triangle$s ABD, ACD,

$\because AB = AC$, AD is common, and $CD = BD$,

$\therefore \angle CAD = \angle BAD$.

Then in $\triangle$s ACE, ABE,

$\because CA = BA$, AE is common,

and $\angle CAE = \angle BAE$,

$\therefore CE = BE$, and $\angle AEC = \angle AEB$;

$\therefore AD$ bisects BC at right angles.

Ex. 9. To any point of the circumference *FEH*.

Ex. 10. Produce *AB*, *AC* two sides of $\triangle$ *ABC* to *D* and *E*, and let $\angle$ *DBC* = $\angle$ *ECB*.

In *CE* or *CE* produced take *CF*=*BD*, and join *DC*, *FB*, *FD*.

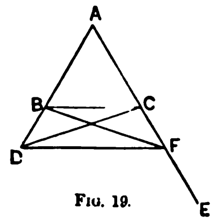


Fig. 19.

Then $\because$ *DB*=*FC*, and *BC* is common,

and $\angle$ *DBC* = $\angle$ *FCB*,

$\therefore$ *DC*=*FB*, and $\angle$ *BDC* = $\angle$ *CFB*,

and $\angle$ *BCD* = $\angle$ *CBF*.

From $\angle$ *CBD* take $\angle$ *CBF*, and from $\angle$ *BCF* take $\angle$ *BCD*,

then $\angle$ *FBD* = $\angle$ *DCF*.

Then $\because$ *FB*=*DC*, and *BD*=*CF*, and $\angle$ *FBD* = $\angle$ *DCF*,

$\therefore$ $\angle$ *BDF* = $\angle$ *CFD*, that is, $\angle$ *ADF* = $\angle$ *AFD*;

$\therefore$ *AD*=*AF*, and $\therefore$ *AB*=*AC*.

Ex. 11. Since *AC*=*AD*, $\therefore$ $\angle$ *ADC* = $\angle$ *ACD*, and $\therefore$ $\angle$ *ADC* is greater than $\angle$ *BCD*; much more is $\angle$ *BDC* greater than $\angle$ *BCD*.

Hence *BC* cannot be equal to *BD*, for then $\angle$ *BCD* would be equal to $\angle$ *BDC*.

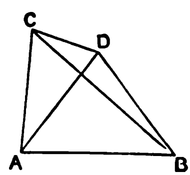


Fig. 20.

Ex. 12. In $\triangle$s *BAE*, *DAC*,

$\because$ *BA*=*DA*, and *AE*=*AC*,

and $\angle$ *BAE* = $\angle$ *DAC*,

$\therefore$ $\angle$ *BEA* = $\angle$ *DCA*=a rt. angle.

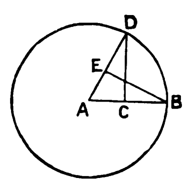


Fig. 21.

Page 25.

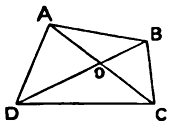


FIG. 22.

EXERCISE. Let $ABCD$ be the quadrilateral figure.
Then $\angle$s AOB, AOD together $=$ two rt. $\angle$s,
and $\angle$s BOC, COD together $=$ two rt. $\angle$s;
$\therefore$ the four $\angle$s at O together $=$ four rt. $\angle$s.

Page 26.

EXERCISE. The words *upon the opposite sides of it* are necessary, because without them the words *the adjacent angles* would have no meaning.

Page 27.

EXERCISE 1. Let EF be the bisector of $\angle AED$, and EG be the bisector of $\angle BEC$.

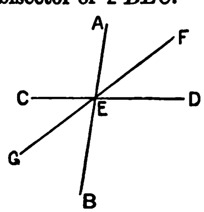


FIG. 23.

Then $\because \angle AED = \angle BEC$,
$\therefore \angle CEG = \angle FED$ (Ax. VII.)
$\therefore$ sum of $\angle$s AEG, AEF
$=$ sum of $\angle$s AEC, CEG, AEF,
$=$ sum of $\angle$s AEC, FED, AEF,
$=$ sum of $\angle$s AEC, AED,
$=$ two rt. $\angle$s;
$\therefore$ EF and EG are in the same st. line.

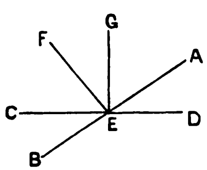


FIG. 24.

Ex. 2. Draw $EF \perp$ to AE, and $EG \perp$ to CE.
Then sum of $\angle$s GEF, GEA $=$ a rt. $\angle$,
$=$ sum of $\angle$s AED, GEA;
$\therefore \angle GEF = \angle AED$.

Ex. 3. In $\triangle$s AED, BEC,

$\because AE=BE$, and $ED=EC$,

and $\angle AED = \angle BEC$,

$\therefore$ the $\triangle$s are equal in all respects.

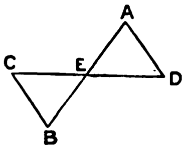


FIG. 25.

Page 29.

EXERCISE 1. If it be possible let AB, AC, AD be three equal st. lines drawn from A to meet the st. line MN.

Then $\because AB=AD$,

$\therefore \angle ABD=ADB$.

Now $\angle ACD$ is greater than $\angle ABD$;

$\therefore \angle ACD$ is greater than $\angle ADB$, that is, $\angle ADC$;

$\therefore AD$ is greater than AC, which is contrary to the hypothesis.

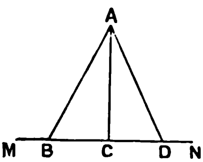


FIG. 26.

Ex. 2. Let FC make with BD the $\angle$s FCB, FCD, of which FCB is obtuse and FCD acute.

From F draw $FE \perp$ to BD: FE must fall on the side of the acute angle.

For if it fell otherwise, as FG, then would $\angle FCE$, an acute angle, be greater than the interior opposite $\angle FGC$, a right angle, which is absurd.

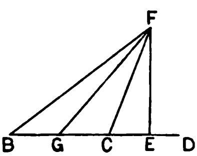


FIG. 27.

Page 31.

EXERCISE. In the $\triangle ABD$ let $\angle ABD = \angle ADB$. Then must $AB=AD$. For if not, let AB be greater than AD. Then will $\angle ADB$ be greater than $\angle ABD$, which is contrary to the hypothesis. Similarly, it may be shown that AB is not less than AD.

$\therefore AB=AD$.

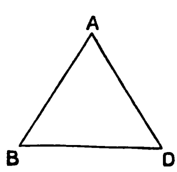


FIG 28.

Page 32.

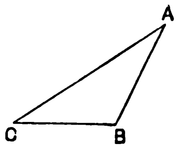


FIG. 29.

EXERCISE 1. In the $\triangle ABC$ let $\angle ABC$ be obtuse.

Then each of the other angles must be acute, and $\therefore$ AC must be greater than either of the other sides.

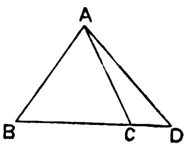


FIG. 30.

Ex. 2. Since $\angle ACB$ is greater than $\angle ADC$,
$\therefore$ $\angle ABC$ is greater than $\angle ADC$,
that is, $\angle ABD$ is greater than $\angle ADB$,
and $\therefore$ AD is greater than AB.

Ex. 3. Draw AB, AC, AD from A to meet the straight line EF, and let AB be $\perp$ to EF, and let AC be nearer than AD to AB.

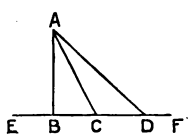


FIG. 31.

Then since $\angle ABC$ is a rt. angle,
$\therefore$ $\angle ACB$ is an acute angle,
$\therefore$ AC is greater than AB.
Also, since $\angle ACD$ is greater than $\angle ABC$,
$\therefore$ $\angle ACD$ is an obtuse angle,
and $\therefore$ $\angle ADC$ is an acute angle,
and $\therefore$ AD is greater than AC.

Page 33.

EXERCISE 1. Let $ABCD$ be a quadrilateral; and join AC.

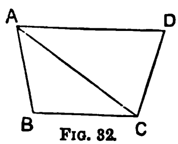


FIG. 32.

Then $\because$ AB, BC together are greater than AC,
$\therefore$ AB, BC, CD together are greater than AC, CD together.

But AC, CD together are greater than AD;
$\therefore$ AB, BC, CD together are greater than AD.

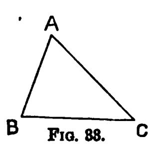


FIG. 33.

Ex. 2. Let ABC be any triangle.
Then AB, AC together are greater than BC.
Take from each AB.
Then AC is greater than the difference between BC and AB.

Ex. 3. Let O be any pt. within, or outside, the quadrilateral $ABCD$.

Then $\because$ OA, OB are together greater than AB,
and OB, OC „ than BC,
and OC, OD „ than CD,
and OD, OA „ than DA;

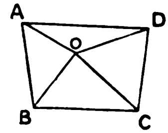


FIG. 34.

$\therefore$ twice the sum of OA, OB, OC, OD is greater than sum of AB, BC, CD, DA;

$\therefore$ sum of OA, OB, OC, OD is greater than half the sum of AB, BC, CD, DA.

Ex. 4. Bisect BC, a side of the $\triangle ABC$ in D, join AD, and produce it to E, making $DE = DA$.

Join EC.

Then $\because$ $BD = CD$, and $DA = DE$, and $\angle BDA = \angle CDE$,

$$\therefore AB = CE.$$

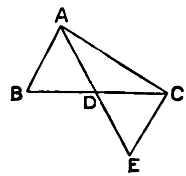


FIG. 35.

Now sum of AC, CE is greater than AE,

$\therefore$ sum of AC, AB is greater than AE, that is, than twice AD.

Page 34.

EXERCISE 1. Produce AD to meet BC in F, and join DB.

Then sum of AC, CF is greater than AF.

$\therefore$ sum of AC, CF, FB is greater than sum of AF, FB,

$\therefore$ sum of AC, CB is greater than sum of AD, DF, FB.

But sum of DF, FB is greater than sum of DE, EB;

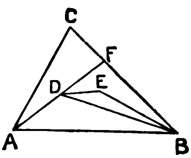


FIG. 36.

$\therefore$ sum of AC, CB is greater than sum of AD, DE, EB.

Ex. 2. Let O be any point within the $\triangle ABC$. Then—

$\because$ sum of AB, BC is greater than sum of OA, OC,
and sum of BC, CA „ than sum of OB, OA,
and sum of CA, AB „ than sum of OB, OC;

$\therefore$ twice sum of AB, BC, CA is greater than twice sum of OA, OB, OC;

$\therefore$ sum of AB, BC, CA is greater than sum of OA, OB, OC.

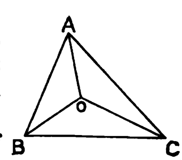


FIG. 37.

Next, ∵ sum of OA, OB is greater than AB,
and sum of OB, OC „ than BC,
and sum of OC, OA „ than CA;
∴ twice sum of OA, OB, OC is greater than sum of AB, BC, CA;
and ∴ sum of OA, OB, OC is greater than half the sum of AB, BC, CA.

Page 35.

Exercise. Take A and C two equal st. lines, and take B a st. line equal to one-half of either A or C, and then proceed as in the Proposition.

Page 41.

Miscellaneous Exercises.

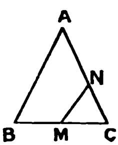


Fig. 38.

1. Difference between MB and MN is equal to difference between MC and MN, and is therefore less than CN,

that is, less than difference between AC, AN,
that is, less than difference between AB, AN.

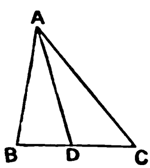


Fig. 39

2. Since $\angle ADB$ is greater than $\angle CAD$,
∴ $\angle ADB$ is greater than $\angle BAD$;
∴ AB is greater than BD.
Again, since $\angle ADC$ is greater than $\angle BAD$,
∴ $\angle ADC$ is greater than $\angle DAC$;
∴ AC is greater than CD.

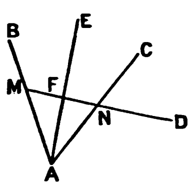


Fig. 40.

3. Draw AE bisecting $\angle BAC$.
Draw $DF \perp$ to AE, meeting AB and AC in M and N, and meeting AE in F.
Then $\angle FAM = \angle FAN$, and $\angle AFM = \angle AFN$, and AF is common to the $\triangle$s AFM, AFN,
∴ $AM = AN$.

4. The construction is the same as in Ex. 3, and then $\angle FMA = \angle FNA$.

5. Draw AD bisecting $\angle BAC$, and AE bisecting $\angle BAG$.

Then $\because$ sum of $\angle$s BAG, BAC = two rt. $\angle$s,
and $\angle BAE$ = half of $\angle BAG$,
and $\angle BAD$ = half of $\angle BAC$,
$\therefore$ sum of $\angle$s BAE, BAD = a rt. $\angle$,
that is, $\angle EAD$ is a rt. $\angle$.

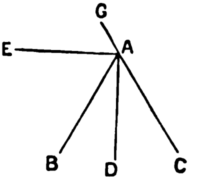


FIG. 41.

6. Let ABC be a triangle, having side AC greater than AB.

Let AD, the bisector of $\angle BAC$, meet BC in D, and let AE bisect BC in E.

From AC cut off $AF = AB$, and join FD.

Then $\because AB = AF$, and AD is common, and $\angle BAD = \angle FAD$,
$\therefore \angle ADF = \angle ADB$,
and $\therefore \angle ADE$ is greater than $\angle ADB$,
$\therefore \angle ADE$ is greater than $\angle AED$,
$\therefore AE$ is greater than AD.

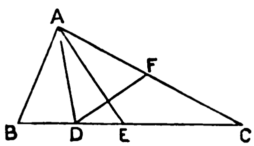


FIG. 42.

7. From any one of the angular points of the $\triangle ABC$, as A, draw AD to meet BC in D.

Then $\angle ADC$ is greater than $\angle ABD$, and $\angle ADB$ is greater than $\angle ACD$;
$\therefore$ sum of $\angle$s ADC, ADB is greater than sum of $\angle$s ABD, ACD;
$\therefore$ sum of $\angle$s ABD, ACD is less than two rt. $\angle$s,
that is, sum of $\angle$s ABC, ACB is less than two rt. $\angle$s.

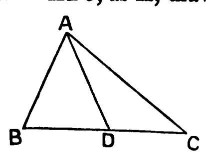


FIG. 43.

8. Since $AB = AD$, and AE is common, and $\angle BAE = \angle DAE$;
$\therefore \angle ABE = \angle ADE$.
But $\angle ADE$ is greater than $\angle ACB$;
$\therefore \angle ABE$ is greater than $\angle ACB$,
that is, $\angle ABC$ is greater than $\angle ACB$.

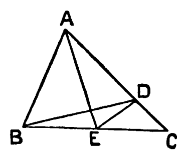


FIG. 44.

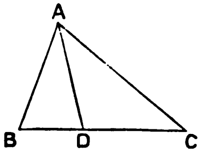


FIG. 45.

9. Let AD be the bisector of $\angle BAC$.

Then, as in Ex. 2, we can show that AB is greater than BD, and that AC is greater than CD;

$\therefore AB, AC$ together are greater than BC.

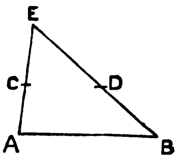


FIG. 46.

10. Let AB be the given side.

At A make $\angle CAB$ = one of the given $\angle$s.

At B make $\angle DBA$ = the other given $\angle$;

AC and BD being drawn so as to lie on the same side of AB.

Then AC, BD will, if produced, meet in some pt. E. (POST. 6.)

$\therefore ABE$ will be the required $\triangle$.

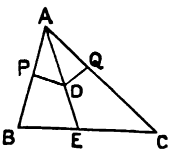


FIG. 47.

11. Let D be any pt. in AE, the bisector of $\angle BAC$

Draw $DP, DQ \perp$ to AB, AC.

Then $\because \angle DAP = \angle DAQ$, and $\angle DPA = \angle DQA$, and AD is common to the $\triangle$s ADP, ADQ,

$\therefore DP = DQ$.

Page 49.

1. Through E, a point equidistant from the parallel lines AB, CD, draw $FEG \perp$ to AB and CD. Then $FE = GE$.

Draw HEK meeting AB in H and CD in K.

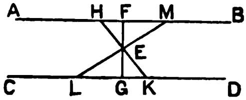


FIG. 48.

Draw LEM meeting CB in L and AB in M.

Then $\because \angle ELG = \angle EMF$, and $\angle LGE = \angle MFE$, and $FE = GE$,

$\therefore FM = LG$.

And $\because \angle EHF = \angle EKG$, and $\angle EFH = \angle EGK$, and $FE = GE$,

$\therefore HF = GK$.

$\therefore$ sum of FM, HF = sum of LG, GK.

$\therefore HM = LK$.

2. Let AD, the bisector of $\angle BAC$, meet BC in D.

Draw DE parallel to AB, meeting AC in E, and DF parallel to AC, meeting AB in F.

Then $\because \angle ADE = \angle DAF$,
and $\angle ADF = \angle DAE$,
$\therefore \angle ADE = \angle ADF$.
Then $\because \angle ADE = \angle ADF$,
and $\angle DAE = \angle DAF$, and AD is common,
$\therefore DE = DF$.

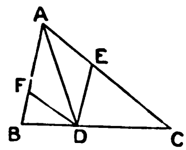


FIG. 49.

3. Let EF, joining the parallel lines AB, CD, be bisected in O and through O draw MON to meet AB in M, and CD in N.

Then $\because \angle MEO = \angle NFO$,
and $\angle EMO = \angle ONF$, and $OE = OF$;
$\therefore MO = NO$.

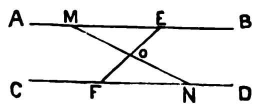


FIG. 50.

Page 50.

EXERCISE 1. Let AB, AC be parallel to DE, DF respectively, and place them so that AC cuts DE in O.

Then $\because AB$ is $\parallel$ to DE,
$\therefore \angle BAC = \angle AOD$;
and $\because AC$ is $\parallel$ to DF,
$\therefore \angle AOD = \angle EDF$;
$\therefore \angle BAC = \angle EDF$.

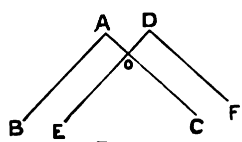


FIG. 51.

Ex. 2. Let AB be $\perp$ to CD.
Join AD and draw $DE \perp$ to AD.
Produce CD to F.
Then shall $\angle BAD = \angle EDF$.
Draw $DG \perp$ to CF. Then DG is $\parallel$ to AB.
Then $\angle GDE$ with $\angle EDF$ = a right $\angle$,
and $\angle GDE$ with $\angle ADG$ = a right $\angle$;
$\therefore \angle EDF = \angle ADG$
$\therefore \angle EDF = \angle BAD$.

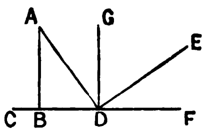


FIG. 52.

Page 51.

Ex. 1. Let A be the given pt., BC the given st. line, E the given $\angle$.

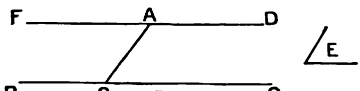


FIG. 53.

Through A draw FD to BC; at A make $\angle\ FAG = \angle\ E$, and let AG meet BC in G.

Then $\angle\ AGC = \angle\ FAG = \angle\ E$.

Ex. 2. Let DE, FG be the two $\parallel$ st. lines, and MN the given line. In FG take $OP=MN$, and with centre O and distance OP

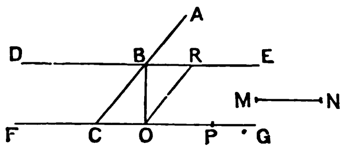


FIG. 54.

descr. a $\odot$ cutting DE in R. Join OR, and draw through A a st. line $ABC \parallel$ to OR, meeting DE in B and FG in C, and join OB.

Then $\because \angle\ BCO = \angle\ DBC$, and $\angle\ DBC = \angle\ BRO$;

$\therefore \angle\ BCO = \angle\ BRO$, and $\angle\ CBO = \angle\ BOR$, and OB is common;

$\therefore BC=RO=MN$.

Page 52.

EXERCISE 1. Since any one of the angles is less than a right angle, the sum of the other two must be greater than a right angle, and therefore the sum of any two is greater than the third.

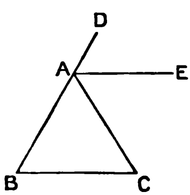


FIG. 55.

Ex. 2. Let ABC be an isosceles $\triangle$, having $AB=AC$.

Produce BA to D, and draw AE bisecting $\angle\ CAD$.

Then $\angle\ CAD$=sum of $\angle$ s ABC, ACB,
=twice $\angle\ ACB$;

$\therefore \angle\ EAC = \angle\ ACB$,
and $\therefore AE$ is parallel to BC.

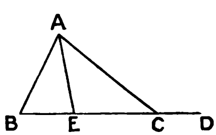


FIG. 56.

Ex. 3. Sum of $\angle$s ABD, ACD,
=sum of $\angle$s ABD, AED, CAE,
=sum of $\angle$s ABD, AED, BAE,
=sum of $\angle$s AED, AED
=twice $\angle\ AED$.

Ex. 4. Let ABC be an isosceles $\triangle$, having $AB = AC$.

Produce BC to D, and let OB, OC, the bisectors of the equal $\angle$s, meet in O.

Then $\angle\ COB$ with $\angle$s OBC, OCB = two rt. $\angle$s;

$\therefore\ \angle\ COB$ with $\angle\ ACB$ = two rt. $\angle$s.

But $\angle\ ACD$ with $\angle\ ACB$ = two rt. $\angle$s;

$\therefore\ \angle\ COB = \angle\ ACD$.

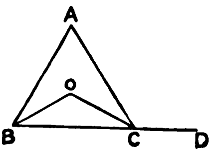


Fig. 57.

Ex. 5. Let ABC be a $\triangle$. Produce BA to D, and let AE, the bisector of $\angle\ CAD$, be $\parallel$ to BC.

Then $\because\ \angle\ DAE$ = interior $\angle\ ABC$,

and $\angle\ EAC$ = alternate $\angle\ ACB$,

$\therefore\ \angle\ ABC = \angle\ ACB$,

and $\therefore\ AB = AC$.

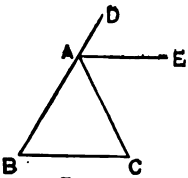


Fig. 58.

Page 54.

Exercise 1. Let $ABCD$ be any quadrilateral, and produce AB to E, BC to F, CD to G, DA to H.

Then, by Cor. II., sum of exterior $\angle$s = four rt. $\angle$s,

and, by Cor. I., sum of interior $\angle$s = four rt. $\angle$s.

$\therefore$ sum of exterior $\angle$s = sum of interior $\angle$s.

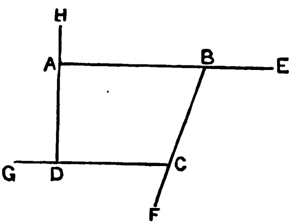


Fig. 59.

Ex. 2. Sum of the six angles with four rt. $\angle$s = twelve rt. $\angle$s,

$\therefore$ sum of the six angles = eight rt. $\angle$s.

Ex. 3. Sum of the five equal angles with four rt. $\angle$s = ten rt. $\angle$s,

$\therefore$ sum of the five equal $\angle$s = six rt. $\angle$s,

$\therefore$ each angle = $\frac{6}{5}$ of a rt. $\angle$.

Ex. 4. Sum of intr. $\angle$s with four rt. $\angle$s = twice as many rt. $\angle$s as the figure has sides.

But sum of intr. $\angle$s in the figure = eight rt. $\angle$s;

$\therefore$ twelve rt. $\angle$s = twice as many rt. $\angle$s as the figure has sides;

$\therefore$ *the* figure has six sides.

Ex. 5. Let n represent the number of sides.
Then $^{7} \times n+4=2n$, or, $7n+16=8n$;
$\therefore n=16$.

Page 56.

1. Let ABC be a $\triangle$, and produce BA, BC, and let OA, OC, the bisectors of the exterior $\angle$s, meet in O. Draw OD, OE, $OF \perp$ s to BA, BC, CA, or to these produced.

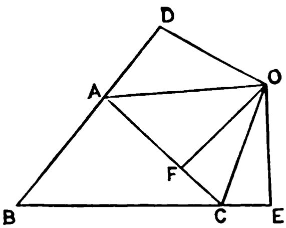


FIG. 60.

Then $\because \angle DAO = \angle FAO$, and $\angle ADO = \angle AFO$, and AO is common;
$\therefore OD=OF$;
and $\because \angle OCE = \angle OCF$, and $\angle OEC = \angle OFC$, and OC is common;
$\therefore OF=OE$.

2. Let BAC be a right angle.

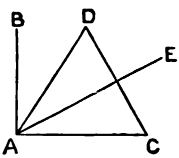


FIG. 61.

On AC describe the equilat. $\triangle ADC$, on the side of AC on which AB stands, and bisect $\angle DAC$ by the st. line AE.

Then $\because \angle DAC=\frac{1}{3}$ of two rt. $\angle$s,
$\therefore \angle EAC=\frac{1}{3}$ of a rt. $\angle$,
$\therefore \angle EAD=\frac{1}{3}$ of a rt. $\angle$,
and $\therefore \angle DAB=\frac{1}{3}$ of a rt. $\angle$.

3. Draw BO, CO the bisectors of $\angle$s ABC, ACB meeting in O. Draw OD, OE, $OF \perp$ to AB, BC, CA, and join AO.

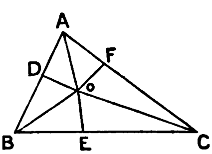


FIG. 62.

Then we have to show that OA bisects $\angle BAC$.

Now $\because \angle OBE = \angle OBD$,
and $\angle OEB = \angle ODB$, and OB is common,
$\therefore OD=OE$;
and $\because \angle OCF = \angle OCE$, and
$\angle OFC = \angle OEC$, and OC is common,
$\therefore OF=OE$, and $\therefore OF=OD$.

Then $\because$ $OF=OD$, and AO is common, and $\angle$s OFA, ODA are rt. $\angle$s,

$\therefore \angle OAD = \angle OAF$. (See Cor. to Prop. E, page 43).

4. Let OD, OE, bisecting AB, BC at rt. $\angle$s meet in O.

Join AO, BO, CO, and draw OF to F the middle point of AC. We have then to show that OF is $\perp$ to AC.

Now $\because$ $BE=CE$, and OE is common, and $\angle OEB = \angle OEC$,

$\therefore OB=OC$,

and $\because$ $BD=AD$, and OD is common, and $\angle ODB = \angle ODA$,

$\therefore OB=OA$, and $\therefore OA=OC$.

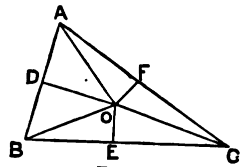


Fig. 63.

Then $\because$ $OA=OC$, and OF is common, and $FA=FC$,

$\therefore \angle OFA = \angle OFC$, and $\therefore OF$ is $\perp$ to AC.

5. In $\triangle ABC$ draw $AD \perp$ to BC, and AE bisecting $\angle BAC$.

Then sum of $\angle$s ABC, $BAE = \angle AEC$,

$=$sum of $\angle$s DAE, ADE,

$=$sum of $\angle$s DAE, ADB,

$=$sum of $\angle$s DAE, DAC, ACB,

$=$sum of $\angle$s DAE, DAE, EAC, ACB.

Now $\angle BAE = \angle EAC$,

$\therefore \angle ABC =$twice $\angle DAE$ with $\angle ACB$;

$\therefore$ difference between $\angle ABC$ and $\angle ACB =$twice $\angle DAE$.

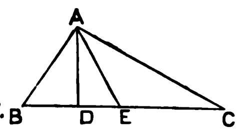


Fig. 64.

6. $\because \angle BDA = \angle EDA$,

and $\angle BAD = \angle EAD$, and AD is common,

$\therefore BD=ED$.

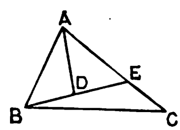


Fig. 65.

7. Let ABC be a rt.-angled $\triangle$, having $\angle BAC$ a rt. $\angle$.

Make $\angle BAD = \angle ABC$;

then $\angle DAC = \angle ACB$.

Then $\because \angle BAD = \angle ABD$, $\therefore AD=DB$;

and $\because \angle DAC = \angle ACD$, $\therefore AD=DC$.

$\therefore ABD$ and ACD are isosceles triangles.

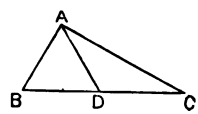


Fig. 66.

8. Produce BA, DC to meet in F.
Draw $EM \parallel$ to CD, meeting FB in M.
In MB make $MG = MF$.
Join GE and produce it to meet FD in H.

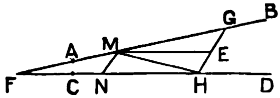


FIG. 67.

Draw $MN \parallel$ to GH, and join MH.
Then $\because \angle GME = \angle MFN$,
and $\angle MGE = \angle FMN$, and $MG = FM$,
$\therefore MN = GE$.
And $\because \angle EMH = \angle MHN$,
and $\angle EHM = \angle NMH$, and MH is common,
$\therefore EH = MN$;
and $\therefore EH = GE$.

9. Let R be the middle pt. of PQ.

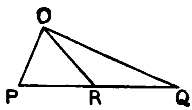


FIG. 68.

(1.) If $OR = PR$, then $\angle ROP = \angle OPR$,
and $OR = QR$, and $\angle ROQ = \angle OQR$;
$\therefore \angle POQ =$ sum of $\angle$s OPQ, OQP;
$\therefore \angle POQ$ is a rt. angle.

(2.) If OR is greater than PR, then $\angle ROP$ is less than $\angle OPR$, and OR is greater than QR, and $\angle ROQ$ is less than $\angle OQR$;
$\therefore \angle POQ$ is less than the sum of $\angle$s OPQ, OQP;
$\therefore \angle POQ$ is an acute angle.

(3.) If OR is less than PR, then $\angle ROP$ is greater than $\angle OPR$, and OR is less than QR, and $\angle ROQ$ is greater than $\angle OQR$;
$\therefore \angle POQ$ is greater than the sum of $\angle$s OPQ, OQP;
$\therefore \angle POQ$ is an obtuse angle.

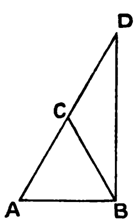


FIG. 69.

10. Let AB be the given st. line.

Describe on AB an equilateral triangle ACB, and produce AC to D, making $CD = AC$.

Join DB, this line shall be $\perp$ to AB.

For BC, the line bisecting AD, is equal to the half of AD;

$\therefore ABD$ is a rt. angle, by Ex. 9.

Page 59.

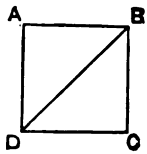


FIG. 70.

1. Let $ABCD$ be a square, and BD a diagonal.

Then $\because AB = AD$, $\therefore \angle ABD = \angle ADB$, and since sum of $\angle$s ABD, ADB, BAD = two rt. $\angle$s, and $\angle BAD$ is a rt. $\angle$,

$\therefore$ each of the $\angle$s ABD, ADB is half a rt. $\angle$.

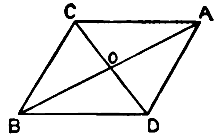


FIG. 71.

2. Let AB, CD bisect each other in O.

Join AD, DB, BC, CA.

Then $\because AO = BO$, and $CO = DO$,
and $\angle AOC = \angle BOD$,
$\therefore \angle OAC = \angle OBD$,
$\therefore AC$ is $\parallel$ to BD.

In the same way it may be shown that BC is $\parallel$ to AD,
$\therefore ADBC$ is a $\square$.

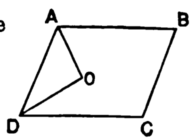


FIG. 72.

3. Let $ABCD$ be a $\square$, and let AO, DO, the bisectors of $\angle$s BAD, CDA meet in O.

Then $\because$ sum of $\angle$s BAD, CDA = two rt. $\angle$s,
$\therefore$ sum of $\angle$s OAD, ODA = a rt. $\angle$,
$\therefore \angle AOD$ is a rt. $\angle$.

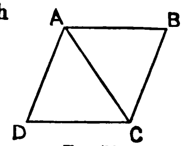


FIG. 73.

4. Let $ABCD$ be a $\square$, and let AC bisect each of the $\angle$s BAD, DCB.

Then $\because \angle BAC = \angle CAD$,
and $\angle BAC = \angle ACD$,
$\therefore \angle CAD = \angle ACD$, and $\therefore AD = CD$.

Therefore all the sides of the $\square$ are equal.

5. Let $ABCD$ be a quadrilateral, having $\angle ABC = \angle ADC$, and $\angle BAD = \angle DCB$.

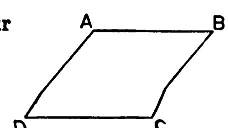


FIG. 74.

Then since the four angles together = four rt. $\angle$s,

$\therefore$ sum of $\angle$s BAD, ADC = two rt. $\angle$s.
and sum of $\angle$s ABC, BAD = two rt. $\angle$s,
$\therefore$ *AB is $\parallel$ to CD,* and AD is $\parallel$ to BC.

6. Let $ABCD$ be a quadrilateral, and let $AB=CD$, and $AD=BC$.

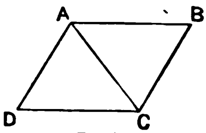


Fig. 75.

Join AC.

Then $\because AB=CD$, and $BC=AD$,

and AC is common,

$\therefore \angle BCA = \angle DAC$, and $\angle BAC = \angle ACD$;

$\therefore AD$ is $\parallel$ to BC, and AB is $\parallel$ to CD.

7. Let $ABCD$ be a rhombus, and let $\angle BAD$ be two-thirds of two rt. angles.

Then $\angle BCD$=two-thirds of two rt. angles. Join AC.

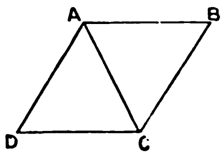


Fig. 76.

Then $\because BA=DA$, and AC is common,

and $BC=DC$,

$\therefore \angle BAC = \angle DAC$, and $\angle ACB = \angle ACD$.

Hence $\angle BAC$=one-third of two rt. angles,

and $\angle ACB$=one-third of two rt. angles,

$\therefore \angle ABC$=one-third of two rt. angles,

and $\triangle ABC$ is equilateral.

Similarly it may be shown that $\triangle ADC$ is equilateral.

Page 60.

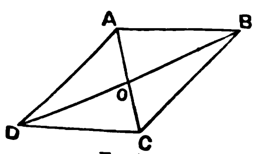


Fig. 77.

Exercise 1. Let the diagonals of the ▱ $ABCD$ meet in O.

Then $\because \angle ABO = \angle CDO$,

and $\angle BAO = \angle DCO$, and $AB=CD$,

$\therefore AO=CO$, and $BO=DO$.

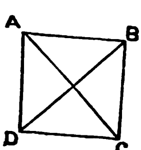


Fig. 78.

Ex. 2. Let $ABCD$ be a rectangle.

Then in the $\triangle$s ABC, DCB,

$\because AB=DC$, and BC is common,

and $\angle ABC = \angle DCB$,

$\therefore AC=BD$.

Page 63.

EXERCISE 1. Draw $PQ \parallel$ to AD and BC.
Then $\because APQD$ is a $\square$, $\therefore \triangle PAD = \triangle PQD$;
and $\because PBCQ$ is a $\square$, $\therefore \triangle PBC = \triangle PQC$;
$\therefore$ sum of $\triangle$s $PAD, PBC = \triangle PDC$.

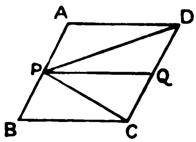


Fig. 79.

Ex. 2. $\triangle ACD = \triangle BCD$, because they are on the same base CD, and between the same parallels.
Take from each $\triangle CED$.
Then $\triangle AEC = \triangle BED$.

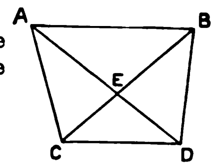


Fig. 80.

Page 64.

EXERCISE 1. Let AD bisect BC in D.
Then $\triangle ABD = \triangle ADC$, because they are on equal bases and between the same parallels, a line through A being assumed parallel to BC.

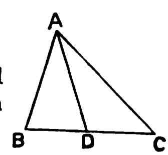


Fig. 81.

Ex. 2. In CA make $CF = BD$ or AE.
Join BF, DF.
Then $\because DB = FC$, and BC is common,
and $\angle DBC = \angle FCB$,
$\therefore \triangle DBC = \triangle FCB$.
Now $\triangle FCB = \triangle ABE$, (I. 38.)
$\therefore \triangle DBC = \triangle ABE$.

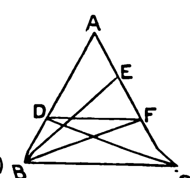


Fig. 82.

Page 65.

Exercise 1. Join CD, and let it meet EP in O.

Then since $\triangle PEB = \triangle ABC$, take from each $\triangle BEC$;

$\therefore \triangle EPC = \triangle ABE$.

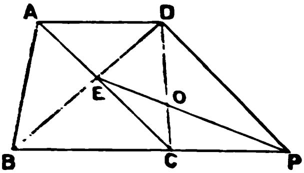


Fig. 33.

Also, since $\triangle ABD = \triangle ACD$. (I. 37.)

take from each $\triangle AED$;

$\therefore \triangle ABE = \triangle EDC$.

$\therefore \triangle EPC = \triangle EDC$.

Take from each $\triangle EOC$;

$\therefore \triangle OCP = \triangle EOD$.

Add to each $\triangle DOP$;

$\therefore \triangle PCD = \triangle PED$.

$\therefore AC$ is $\parallel$ to PD.

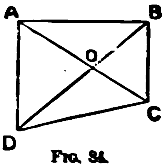


Fig. 34.

Ex. 2. Let the diagonals of the quadrilateral $ABCD$ intersect in O, and let $\triangle AOB = \triangle DOC$.

Add to each $\triangle BOC$.

Then $\triangle ABC = \triangle BDC$,

and $\therefore AD$ is $\parallel$ to BC.

Page 66.

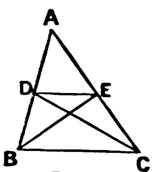


Fig. 35.

Exercise 1. Let D and E be the middle pts. of AB, AC. Join DE.

Then $\triangle CDE = \triangle ADE$, (I. 38.)

and $\triangle BDE = \triangle ADE$; (I. 38.)

$\therefore \triangle CDE = \triangle BDE$.

$\therefore DE$ is $\parallel$ to BC. (I. 39.)

Ex. 2. Let D, E, F be the middle pts. of AB, BC, CA. Join DE, EF, FD.

Then DF is $\parallel$ to BC,
and FE is $\parallel$ to AB,
$\therefore$ $DBEF$ is a $\square$;
$\therefore \triangle DFE = \triangle DBE$.

Now $\triangle ADF = \triangle BDE$, $\because$ FE is $\parallel$ to AB;
and $\triangle FEC = \triangle DBE$, $\because$ DF is $\parallel$ to BC;
$\therefore$ the four triangles are equal.

Also note that $DF = BE =$ the half of BC.

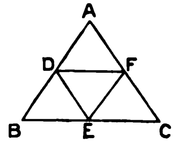


FIG. 86.

Page 67.

EXERCISE 1. Let $ABCD$ be the $\square$, O the pt. without. Join OA, OB, OC, OD. Through O draw $EOF \parallel$ to AB, meeting DA, CB produced in E and F.

Then $\triangle ODC =$ half of $\square$ $EDCF$,
and $\triangle OAB =$ half of $\square$ $EABF$,
$\therefore$ difference between $\triangle ODC$ and $\triangle OAB =$ half of $\square$ $ABCD$.

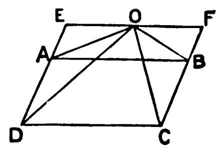


FIG. 87.

Ex. 2. Take O, any pt. within the $\square$ $ABCD$, and join OA, OB, OC, OD.

Through O draw $EOF \parallel$ to AD.

Then $\triangle AOD =$ half of $\square$ $ADFE$,
and $\triangle OBC =$ half of $\square$ $EBCF$,
$\therefore$ sum of $\triangle$s AOD, $OBC =$ half of $\square$ $ABCD$.

Similarly, sum of $\triangle$s AOB, $COD =$ half of $\square$ $ABCD$.

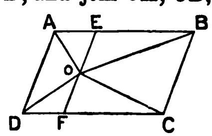


FIG. 88.

Page 68.

EXERCISE 1. Let $ABCD$ be the given $\square$, M the given $\angle$.

Produce AB to E making $BE = AB$.

Make $\angle FAE = \angle M$, and let AF meet DC or DC produced in F, and join FE, BF.

Then $\triangle AFE$ is double of $\triangle ABF$,
and $\square$ $ABCD$ is double of $\triangle ABF$;
$\therefore \triangle AFE = \square ABCD$, and has an $\angle FAE = \angle M$.

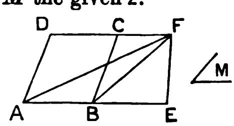


FIG. 89.

Ex. 2. Let ABC be the given $\triangle$.

Draw $AD \parallel$ to BC, bisect BC in O, and with centre O and distance equal to half the sum of BA, AC, describe a circle cutting AD in E, and join OE, and through C draw $CF \parallel$ to OE, meeting AD in F.

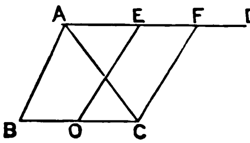


FIG. 90.

Then $EOCF$ is a $\square$, the sum of whose sides is equal to the sum of the sides of $\triangle ABC$,

for sum of OE, CF = sum of AB, AC,

and sum of OC, $EF = BC$.

Also $\square\ EOCF = \triangle ABC$, since $OC = \frac{1}{2} \cdot BC$.

Ex. 3. Let ABC be an isosceles $\triangle$. Draw $AO \perp$ to BC, then AO bisects BC. Complete the rectangle $AOCD$.

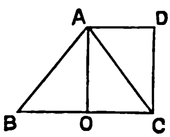


FIG. 91.

Now AC is greater than AO, $\because \angle AOC$ is a rt. $\angle$; and $\therefore AB$ is greater than CD.

Also, AD, OC together $= BC$.

$\therefore$ sum of AB, BC, CA is greater than sum of AO, OC, CD, DA.

Page 69.

EXERCISE 1. If O be not in AC, let it lie on the side of AC nearest to B, and let the line drawn through $O \parallel$ to BC cut AC in P.

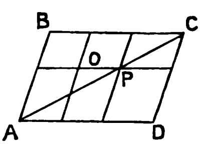


FIG. 92.

Through P draw another line $\parallel$ to CD.

Then $\square\ BP = \square\ PD$; (I. 43.)

$\therefore \square\ OB$ is less than $\square\ OD$, which is contrary to the hypothesis.

Similarly it may be shown that O does not lie on the side of AC nearest to D, and $\therefore O$ will be in AC.

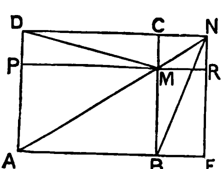


FIG. 92*a*.

Ex. 2. Complete the $\square\ CBFN$.

Draw through M a line $\parallel$ to DN, meeting AD in P, and FN in R.

Then $\square\ DM = \square\ MF$. (I. 43.)

Now $\square\ DM$ = twice $\triangle MDC$; (I. 34.)

and $\square\ MF$ = twice $\triangle MBN$; (I. 41.)

$\therefore \triangle MBN = \triangle MDC$.

Page 72.

Miscellaneous Exercises.

1. Let AC, a diagonal of the quadrilateral $ABCD$, bisect the other diagonal, BD, in O.

Then $\because BO = DO$, $\therefore \triangle AOB = \triangle AOD$;

and $\because BO = DO$, $\therefore \triangle COB = \triangle COD$;

$\therefore$ sum of $\triangle$s AOB, COB = sum of $\triangle$s AOD, COD;

$\therefore \triangle ABC = \triangle ADC$.

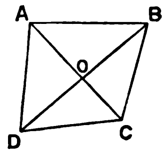


Fig. 93.

2. Let the diagonals of the $\square$ $ABCD$ meet in M.

Take O any pt. in DB, or in DB produced, and join OA, OC.

Then $\because MA = MC$. (Ex. 1, p. 60.)

$\therefore \triangle MAB = \triangle MCB$,

and $\triangle MAO = \triangle MCO$;

$\therefore \triangle AOB = \triangle COB$.

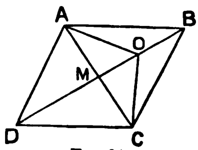


Fig. 94.

3. Let $ABDC$ be a trapezium, having $AB \parallel$ to CD.

Let M, N be the middle pts. of AB, CD, and join MN, AN, BN.

Then $\because AM = BM$, $\therefore \triangle AMN = \triangle BMN$;

and $\because CN = DN$, $\therefore \triangle ANC = \triangle BND$;

$\therefore$ sum of $\triangle$s AMN, ANC = sum of $\triangle$s BMN, BND.

$\therefore$ fig. $ACNM$ = fig. $BDNM$.

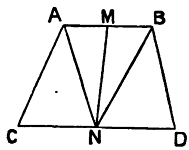


Fig. 95.

4. Sum of $\triangle$s CPD, BPC = fig. $DPBC$

= sum of $\triangle$s CDB, BPD.

Now $\triangle CDB = \frac{1}{2} \square ABCD$,

$= \triangle ABC$,

= sum of $\triangle$s APB, BPC, APC;

$\therefore$ sum of $\triangle$s CPD, BPC = sum of $\triangle$s APB, BPC, APC, BPD;

$\therefore \triangle CPD$ = sum of $\triangle$s APB, APC, BPD;

$\therefore$ difference of $\triangle$s CPD, APB = sum of $\triangle$s APC, BPD.

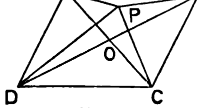


Fig. 96.

5. Let $ABCD$ be a ▱, and let $AC = AB$.

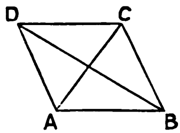


Fig. 97.

Now since each of the equal angles in an isosceles $\triangle$ must be less than a rt. $\angle$, (I. 17.)

$\therefore \angle ABC$ is less than a rt. $\angle$;

$\therefore \angle BAD$ is greater than a rt. $\angle$. (I. 29.)

$\therefore BD$ is greater than AD or AB. (I. 19, Ex. 1.)

$\therefore BD$ is greater than any side of the figure.

6. Let $ABCD$ be a ▱.

Draw EF, FG, GH, HE through A, B, C, D $\parallel$ to the diagonals of ▱ $ABCD$.

Then $\because EF$ and HG are both $\parallel$ to BD;

$\therefore EF$ is $\parallel$ to HG.

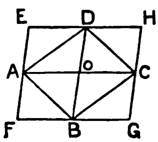


Fig. 98.

Similarly FG is $\parallel$ to EH;

$\therefore EFGH$ is a ▱.

Let O be the intersection of AC and BD.

Then ▱ $AOBF$ = twice $\triangle AOB$,

▱ $BOCG$ = twice $\triangle BOC$,

▱ $CHDO$ = twice $\triangle DOC$,

▱ $DOAE$ = twice $\triangle DOA$;

$\therefore$ ▱ $EFGH$ = twice ▱ $ABCD$.

7. Let ABC, DEF be two $\triangle$s having $AB = DE$, $AC = DF$, and $\angle BAC$ the supplement of $\angle FDE$.

Complete the ▱ $ABCG$, and join AG.

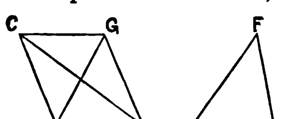


Fig. 99.

Then $\angle BAC$ is the supplement of $\angle ABG$.

$\therefore \angle ABG = \angle FDE$, and $AB = DE$,

and $BG = FD$;

$\therefore \triangle ABG = \triangle EDF$.

But $\triangle ABC = \triangle ABG$;

$\therefore \triangle ABC = \triangle DEF$.

8. Let ABC be the given $\triangle$, and P the given pt. in AC.

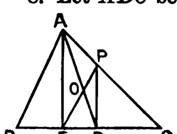


Fig. 100.

Bisect BC in D, join AD, PD, and draw $AE \parallel$ to PD.

Join PE, cutting AD in O.

Then shall PE bisect the $\triangle ABC$.

For $\because AE$ is $\parallel$ to PD, $\therefore \triangle APD = \triangle EPD$; take from each $\triangle POD$; then $\triangle AOP = \triangle EOD$.

Also, $\because BD = CD$, $\therefore \triangle ABD = \triangle ACD$, of which the parts AOP, EOD are equal, and $\therefore$ fig. $ABEO$ = fig. $PODC$;

$\therefore$ sum of $ABEO$ and $\triangle AOP$ = sum of $PODC$ and $\triangle EOD$;

$\therefore$ fig. $ABEP = \triangle PEC$.

9. Since $\triangle AEC = \triangle ABE$ (I. 38.)

$= \triangle EBD$;

$\therefore \triangle ABC = \triangle ADE$.

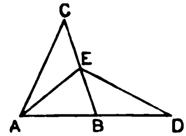


FIG. 101.

10. Take the diagram of I. 43, and join EG, HK: these lines shall be ||.

Join HE, KG.

Then $\because AF = FC$, $\therefore \triangle HEF = \triangle KFG$; add to each $\triangle HKF$.

Then $\triangle HEK = \triangle HGK$;

and $\therefore EG$ is || to HK.

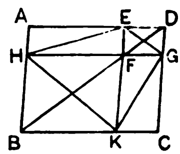


FIG. 102.

Page 73.

EXERCISE 1. Let AB, CD be the two given lines.

Draw $EF = CD$, and from E draw $EG = AB$ and $\perp$ to EF. Complete the rectangle $EFHG$, which will be described as required.

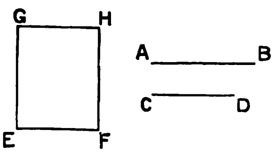


FIG. 103.

Ex. 2. Let $ABCD$, $EFGH$ be squares on equal st. lines AB, EF.

Apply $EFGH$ to $ABCD$, so that E lies on A and EF on AB, then $\because EF = AB$,

F will coincide with B.

And since $\angle DAB = \angle HEF$, EH will fall on AD, and since $EH = AD$, H will coincide with D. Similarly it may be shown that G will coincide with C.

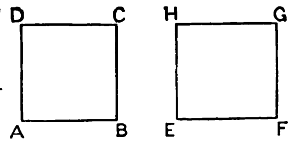


FIG. 104.

$\therefore$ *EFGH coincides* with and is $\therefore$ equal to ABCD.

Ex. 3. Let $ABCD$, $EFGH$ be equal squares (diagram of Ex. 2). Apply $EFGH$ to $ABCD$, so that E coincides with A, and EF falls on AB and EH on AD; then must F coincide with B. For if F falls between A and B, then H falls between A and D, and G will fall inside $ABCD$, and $EFGH$ will be enclosed by $ABCD$, which is impossible. And if F falls on AB produced, then G will fall outside $ABCD$, and $ABCD$ will be enclosed by $EFGH$, which is impossible.

$\therefore$ F will coincide with B, and $\therefore$ $EF=AB$.

Page 75.

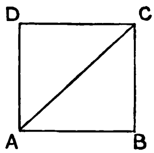


FIG. 105.

EXERCISE 1. Let $ABCD$ be the given square, AC a diagonal.

Then ABC is a right-angled $\triangle$,

and sq. on AC = sum of sqq. on AB, BC,

= twice sq. on AB,

= twice the given square.

Ex. 2. Let A, B, C be three given lines.

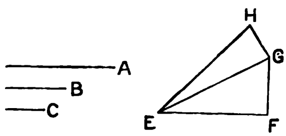


FIG. 106.

Take $EF=A$, and from F draw $FG=B$, and $\perp$ to EF.

Join EG. Then sq. on EG = sum of sqq. on A, B.

From G draw $GH=C$, and $\perp$ to EG. Join EH.

Then sq. on EH = sum of sqq. on EG, GH.

= sum of sqq. on A, B, C.

Ex. 3. Let ABC be a triangle, having $\angle$ ACB equal to the sum of the other two angles.

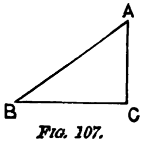


FIG. 107.

Then $\angle$ ACB is a rt. $\angle$. (I. 32.)

Also since the sum of the squares of 4 and 3 is $16+9$, or 25, and since the square of 5 is 25, AB must contain five parts each equal to one of the equal parts into which BC and CA are divisible.

Ex. 4. Sum of sqq. on AC, BC=sq. on AB,
=sq. on DE,
=sum of sqq. on DF, EF.
But sq. on AC=sq. on DF;
$\therefore$ sq. on BC=sq. on EF;
$\therefore$ $BC=EF$, and $\therefore$ the triangles are equal in all respects.

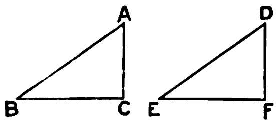


Fig. 108.

Ex. 5. Let AB be the given st. line. Draw $AC \perp$ to AB, and make $AC=AB$. Join CB. Bisect $\angle BCA$ by CD meeting AB in D, and draw $DE \perp$ to BC.

Then $\because$ $\angle ACD = \angle ECD$, and $\angle CAD = \angle CED$, and CD is common,

$$\therefore AD=DE.$$

Also, since $\angle EBD$=half a rt. $\angle = \angle EDB$,

$$\therefore EB=DE.$$

Then sq. on DB=sum of sqq. on DE, BE,
=twice sq. on DE,
=twice sq. on DA.

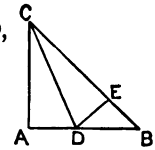


Fig. 109.

Ex. 6. Let ABC be a right-angled $\triangle$, having $\angle ABC$ a rt. $\angle$.
Draw AD to meet BC in D.
Then sum of sqq. on BC, AD
=sum of sqq. on BC, AB, BD,
=sum of sqq. on AC, BD.

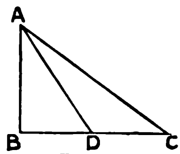


Fig. 110.

Ex. 7. Let ABC be any triangle, and let AD be drawn at rt. angles to BC.
Then $\because$ sq. on AC=sum of sqq. on CD, AD,
and sq. on AB=sum of sqq. on BD, AD;
difference between sqq. on AC, AB
=difference between sqq. on CD, BD.

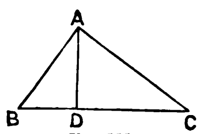


Fig. 111.

END OF BOOK I.

BOOK II.

Page 78.

EXERCISE. Let AB be divided into any number of parts in E, F, G; and let AC be divided into any number of parts in H, K.

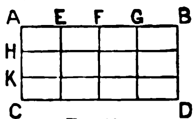


FIG. 112.

Place AC to make a rt. $\angle$ with AB.
Through E, F, G, B draw lines $\parallel$ to AC.
Through H, K, C draw lines $\parallel$ to AB.
Then the proof is the same as that in PROP. I.

Page 79.

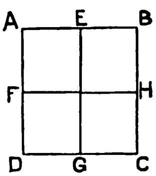


FIG. 113.

On AB describe a square $ABCD$.
Bisect AB, AD in the pts. E and F.
Draw $EG \parallel$ to AD, and $FH \parallel$ to AB.
Then the square $ABCD$ is divided into four squares, each of which is equal to the square on AE.

$\therefore$ sq. on AB = four times sq. on AE.

Page 81.

EXERCISE. Let ABC be the $\triangle$, BAC the rt. $\angle$.
Draw $AD \perp$ to BC.

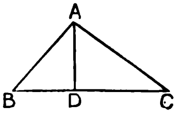


FIG. 114.

Then sq. on BD with sq. on CD with twice rect. BD, DC = sq. on BC (II. 4.)
= sq. on BA with sq. on CA,
= sum of sqq. on BD, DA, DA, CD;
$\therefore$ twice rect. BD, DC = twice sq. on DA;
$\therefore$ rect. BD, DC = sq. on DA.

Page 84.

EXERCISE. In PROP. V. AD is the sum of AC, CD,
DB is the difference of AC, CD;
$\therefore$ rect. AD, DB = difference of sqq. on AC, CD,
or, rect. AD, DB with sq. on CD = sq. on AC.

In PROP. VI. AD is the sum of CD, AC;
$\therefore$ AD is the sum of CD, CB,
DB is the difference of CD, CB;
$\therefore$ rect. AD, DB = difference of sqq. on CD, CB;
or, rect. AD, DB with sq. on CB = sq. on CD.

Page 85.

EXERCISE. Since $\angle$ CGB = half a rt. $\angle$,
and $\angle$ CGH = a rt. $\angle$,
and $\angle$ HGD = half a rt. $\angle$,
$\therefore$ sum of $\angle$s CGB, CGH, HGD = two rt. $\angle$s;
$\therefore$ BGD is a st. line.

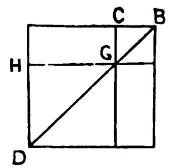


FIG. 115.

Page 89.

EXERCISE. By II. 7
sq. on AB with sq. on BH = twice rect. AB, BH with sq. on AH,
= twice sq. on AH with sq. on AH,
= three times sq. on AH.

Page 90.

EXERCISE. Let $ABCD$ be a trapezium, having $AB \parallel$ to CD.
Draw AE, BF $\perp$ to CD, or CD produced.

Then sq. on AC = sq. on AD with sq. on CD, with twice rect. CD, DE; and sq. on BD = sq. on BC with sq. on CD, with twice rect. CD, CF,

$\therefore$ sqq. on AC, BD = sqq. on AD, BC with twice sq. on CD with twice rect. CD, DE, *with twice rect.* CD, CF.

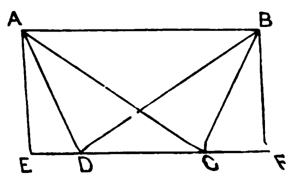


FIG. 116.

Again, rect. AB, CD=rect. EF, CD=rect. CD, DE, with rect. CD, CD with rect. CD, CF. (II. 1.)

∴ twice rect. AB, CD=twice rect. CD, DE with twice sq. on CD with twice rect. CD, CF;

∴ sqq. on AC, BD=sqq. on AD, BC with twice rect. AB, CD.

Note.—The angles at C and D have been drawn as *obtuse* angles. If either or both be *acute* angles, the proof is similar, but it will depend on Prop. XIII. with, or instead of, Prop. XII.

Page 91.

Exercise. Let ABC be any △, AE the ⊥ from A on BC, AD the line drawn from A to bisect BC in D.

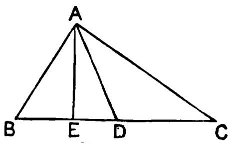


Fig. 117.

Then sq. on AB=sqq. on BD, AD diminished by twice rect. BD, DE; and sq. on AC=sqq. on CD, AD increased by twice rect. CD, DE;

∴, observing that $BD=CD$, sum of sqq. on AB, AC=twice sum of sqq. on BD, AD.

N.B.—This theorem is of great importance, and it will be frequently referred to.

Page 93.

Miscellaneous Exercises on Book II.

1. Let ∠ BAC be a rt. ∠, and let AD be ⊥ to BC.

Then, since ∠ ABC is an acute ∠, sq. on AC with twice rect. BC, BD=sqq. on AB, BC (II. 13.)

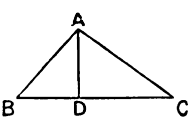


Fig. 118.

=sqq. on AB, AB, AC;

∴ twice rect. BC, BD=twice sq. on AB;

∴ rect. BC, BD=sq. on AB.

Similarly it may be shown that rect. BC, CD =sq. on AC.

2. Let $ABCD$ be a ▱, and draw AE, BF ⊥ to DC or DC produced. Then $DE=CF$.

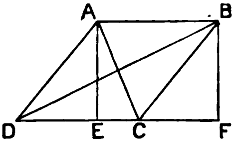


FIG. 119.

Now sq. on AC = sqq. on AD, DC diminished by twice rect. CD, DE, and sq. on BD=sqq. on BC, DC increased by twice rect. CD, CF;

∴, observing that $AB=DC$,

sum of sqq. on AC, BD=sum of sqq. on AB, BC, CD, DA.

3. Let $ABCD$ be a rectangle, and let the diagonals intersect in P. Then they bisect each other, and AP, BP, CP, DP are all equal. (I. 34, Ex. 1 and 2.)

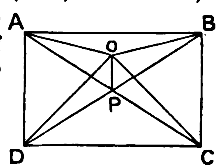


FIG. 120.

Join OP. Then, as is proved in the Ex. to II. 13, sum of sqq. on AO, OC=twice sum of sqq. on AP, OP; and sum of sqq. on OB, OD =twice sum of sqq. on DP, OP

∴, since $AP=DP$,

sum of sqq. on AO, OC=sum of sqq. on OB, OD.

4. In the ▱ $ABCD$, let $BD=DC=AB$.

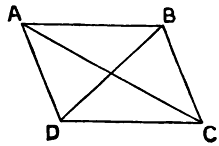


FIG. 121.

Now sum of sqq. on AC, DB

=sum of sqq. on AB, BC, CD, DA (Ex. 2.)

=sum of sqq. on DB, BC, DB, BC;

∴ sq. on AC=sq. on DB with twice sq. on BC;

∴ sq. on DB is less than sq. on AC by twice sq. on BC.

5. Let D be the side of the given square.

Draw $BE=D$ at rt. ∠s to AB.

With centre A and distance AE describe a ⊙, and let AB produced meet the ○ce in C.

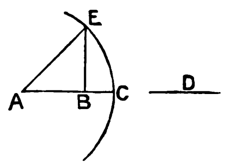


FIG. 122.

Then rect. contained by the sum and difference of AB, AC,

=difference of sqq. on AC, AB, (II. B.)

=difference of sqq. on AE, AB,

=sq. on BE=sq. on D.

6. Let $ABCD$ be a quadrilateral, and let O, P be the middle pts. of its diagonals. Join OP, BP, DP.

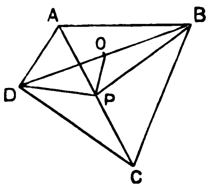


FIG. 123.

Then by Ex. to II. 13,
sum of sqq. on AB, BC = twice sum of sqq. on BP, CP;
and sum of sqq. on CD, DA = twice sum of sqq. on DP, CP;
∴ sum of sqq. on the four sides = twice sum of sqq. on BP, DP with four times sq. on CP.
But sum of sqq. on BP, DP = twice sum of sqq. on BO, OP; (II. 13, Ex.)
∴ sum of sqq. on the four sides = four times sum of sqq. on BO, OP, CP.
Also, sq. on AC = four times sq. on CP; (II. 2, Ex.)
and sq. on BD = four times sq. on BO; (II. 2, Ex.)
∴ sum of sqq. on the four sides = sum of sqq. on diagonals with four times sq. on OP.

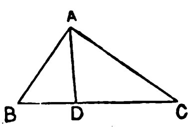


FIG. 124.

7. Let ABC be a △, and AD the ⊥ from A on BC, and let sq. on AD = rect. BD, DC.
Then sq. on BC = sum of sqq. on BD, DC with twice rect. BD, DC,
= sum of sqq. on BD, DC, DA, DA,
= sum of sqq. on BA, AC;
∴ ∠ BAC is a rt. angle. (I. 48.)

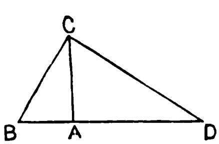


FIG. 125.

8. Let AB, AC be the given lines. Place them so as to be at right angles to each other. Join BC.
Draw CD ⊥ to BC, meeting BA produced in D.
Then, by Ex. to II. 4,
rect. BA, AD = sq. on AC.

9. Divide AB in any points C, D.
On AB describe the sq. $ABEF$.
In AF take $AG = AC$, and $GH = CD$, then $HF = BD$.

Divide AE into nine rectangles by drawing lines from C, D || to AF, and lines from G, H || to AB.

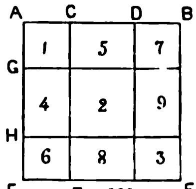


Fig. 126.

Then 1, 2, 3 are the squares on AC, CD, DB,
4, 5 are the rect. AC, CD,
6, 7 are the rect. AC, DB,
8, 9 are the rect. CD, DB;

$\therefore$ sq. on AB=sum of sqq. on AC, CD, DB with twice the rectangle contained by the parts, taken two and two.

10. Let $ABCD$ be a quadrilateral, and E, F, G, H the middle points of its sides.

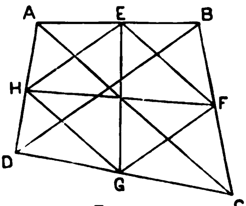


Fig 127.

Then $\because$ EF joins the middle pts. of AB, CB.

$\therefore$ EF is || to AC, and AC=twice EF. (I. 40, Ex. 1.)

Similarly, HG is || to AC, and EH, FG are || to BD, and $\therefore$ $EFGH$ is a $\square$.

Now sq. on AC=four times sq. on EF. (II. 2, Ex.)

$\therefore$ sum of sqq. on AC, BD=four times sum of sqq. on EF, FG,
=twice sum of sqq. on EF, FG, GH, HE,
=twice sum of sqq. on EG, FH.
(By Ex. 2.)

11. Let ABC be a $\triangle$, D, E, F the middle points of BC, CA, AB.

Then sum of sqq. on AB, AC=twice sum of sqq. on AD, BD,
sum of sqq. on AC, CB=twice sum of sqq. on CF, BF,
sum of sqq. on CB, AB=twice sum of sqq. on BE, AE;

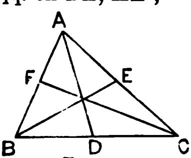


Fig. 128.

$\therefore$ twice sum of sqq. on AB, AC, CB=twice sum of sqq. on AD, CF, BE with twice sum of sqq. on BD, BF, AE;

$\therefore$ four times sum of sqq. on AB, AC, CB =four times sum of sqq. on AD, CF, BE with four times sum of sqq. on BD, BF, AE;

$\therefore$ four times sum of sqq. on AB, AC, CB=four times sum of sqq. on AD, CF, BE with sum of sqq. on CB, AB, AC;

$\therefore$ three times sum of sqq. on AB, AC, CB=four times sum of sqq. on AD, CF, BE.

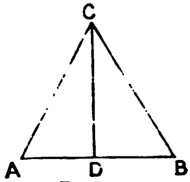


Fig. 129.

12. Sum of sqq. on CD, DA = sq. on CA,
= sq. on AB,
= sum of sqq. on AD, DB with twice rect. AD, DB;
∴ sq. on CD = sq. on DB with twice rect. AD, DB.

13. Since CD bisects AE,

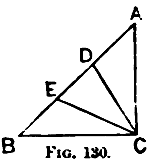


Fig. 130.

sum of sqq. on AC, CE = twice sum of sqq. on CD, DE;
and since CE bisects BD,
sum of sqq. on BC, CD = twice sum of sqq. on CE, DE.
∴ sum of sqq. on AC, BC, CE, CD = twice sum of sqq. on CD, CE, DE, DE;
∴, observing that sq. on AB = sum of sqq. on AC, BC,
sq. on AB = sum of sqq. CD, CE, DE with three times sq. on DE.

Now since DE is one-third of AB, the sq. on DE is one-ninth of sq. on AB;

∴ two-thirds of sq. on AB = sum of sqq. on CD, CE, DE.

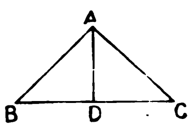


Fig. 131.

14. Let ABC be an isosceles right-angled △, having $BA = AC$, and ∠ BAC the rt. ∠.

Draw AD ⊥ to BC.

Then AD bisects ∠ BAC and also BC.

∴ ∠ DAC = ∠ ACD,

and ∴ $DC = DA$.

∴ sq. on BC = four times sq. on DC = four times sq. on DA.

15. Let AB be the given st. line, and M the other line.

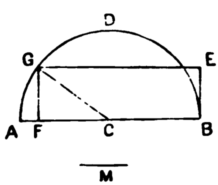


Fig. 132.

Bisect AB in C, and on AB describe the semicircle ADB.

Draw $BE = M$, and ⊥ to AB; and draw EG ∥ to BA to meet the semicircle in G; and draw GF ∥ to EB.

Then rect. AF, FB with sq. on FC
= sq. on CB (II. 5.)
= sq. on CG,
= sum of sqq. on CF, FG.

∴ rect. AF, FB = sq. on FG,
= sq. on M.

Loci on page 104.

(1.) A circle described with the given pt. as the centre, and the given distance as the radius.

(2.) A straight line parallel to the given line.

(3.) A straight line parallel to the given line.

(4.) A straight line bisecting the angle.

(5.) A circle described with the centre of the given circle as its centre, and with a radius equal to the sum of the radius of the given circle and a straight line equal to the given distance.

(6.) Two straight lines bisecting the vertically opposite angles.

Page 116.

Miscellaneous Exercises on Books I. and II.

1. Let AB, CD intersect at right angles in O.

Then $\because$ $AO=OB$, and OC is common, and $\angle AOC = \angle BOC$,

$\therefore$ $AC=CB$; and similarly it may be shown that $AC=AD=DB=CB$.

Again $\because$ $OA=OC$, $\therefore$ $\angle OCA = \angle OAC$;

$\therefore$ $\angle OAC$ is half a rt. $\angle$.

Similarly, $\angle OAD$ is half a rt. $\angle$,

$\therefore$ $\angle CAD$ is a rt. angle; and $\therefore$ $ACBD$ is a square.

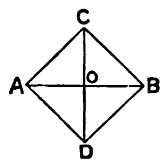


Fig. 133.

2. Let $ABCD$ be a ▱, and P a point in AB.

Bisect BD in F; join PF, and produce it to meet DC in E.

Then $\because$ $\angle FPB = \angle FED$,

and $\angle FBP = \angle FDE$, and $FB=FD$,

$\therefore$ $\triangle PFB = \triangle EFD$.

Also $\triangle ABD = \triangle BDC$.

$\therefore$, by subtraction, fig. $APFD$=fig. $BFEC$;

$\therefore$, adding the equal $\triangle$s FDE, FBP,

fig. $APED$=fig. BPEC.

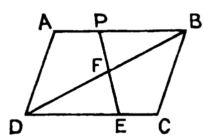


Fig. 134.

3. Produce DC to E, making $CE=DE$, and join EF.

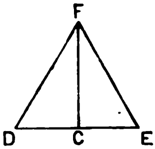


FIG. 135.

Then $\because DC=EC$, and CF is common,
and $\angle FCD = \angle FCE$,
$\therefore FD=FE$.
Now $\angle FDE$ is $\frac{2}{3}$ of a rt. $\angle$,
$\therefore \angle FED$ is $\frac{2}{3}$ of a rt. $\angle$,
and $\therefore \angle DFE$ is $\frac{2}{3}$ of a rt. $\angle$,
$\therefore FDE$ is an equilateral $\triangle$;
$\therefore FD=DE=$twice DC.

4. Let AD, BE, CG the $\perp$s on the sides meet in F, which is proved on p. 56, Ex. 4.

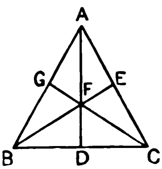


FIG. 136.

Then, since $\angle AFB$ is an obtuse $\angle$,
sq. on $AB=$sum of sqq. on AF, FB with twice rect. AF, FD.
Now $AF=BF=CF$; and $BF=$twice FD; (Ex. 3.)
$\therefore$ sq. on $AB=$sum of sqq. on AF, AF with twice half the sq. on AF.
$=$three times sq. on AF.

5. Let ABC be the given $\triangle$, and BC the side to which each side of the rhombus shall be equal.

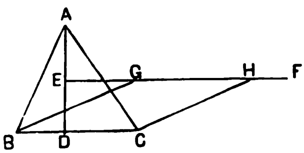


FIG. 137.

Draw $AD \perp$ to BC: bisect AD in E: draw $EF \parallel$ to BC, and with centre B and distance BC describe a $\odot$ cutting EF in G.
Join GB and draw $CH \parallel$ to GB: then $GBCH$ is a rhombus, and area of $GBCH=$rect. ED, $BC=\frac{1}{2}$ rect. AD, $BC=$area of $\triangle ABC$.

Note.—The problem is impossible if BC be less than ED.

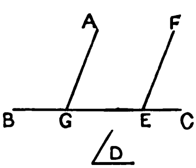


FIG. 138.

6. Let A be the given point, BC the given st. line, D the given angle.
Take E any point in BC, and at E make $\angle FEC = \angle D$. Through A draw $AG \parallel$ to FE, meeting BC in G.
Then $\angle AGC = \angle FEC = \angle D$.

7. Let A, B be the given pts.; DE the given line.

Draw $AD \perp$ to DE, and produce AD to C, making $DC=DA$. Join BC cutting DE in O. Join AO, and from any other pt. P in DE draw AP, BP. Then shall AO, OB be together less than AP, PB together.

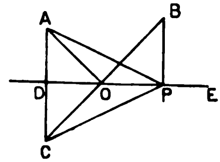


Fig. 139.

For $\because AD=CD$, and OD is common,

and $\angle ADO = \angle CDO$,

$\therefore AO=CO$; and similarly $AP=CP$.

Also, $\angle AOD = \angle COD = \angle BOP$.

Now sum of AO, OB=sum of CO, OB

$=CB$;

and sum of AP, PB=sum of CP, PB,

which is greater than CB;

$\therefore$ sum of AP, PB is greater than sum of AO, OB.

8. Let OP cut AB in M, and let OQ cut CD in N.

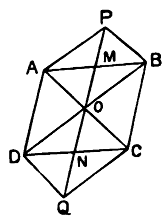


Fig. 140.

Then $\angle QOD$ is supplement of $\angle QOB$.

Now the diagonals of a $\square$ bisect each other,

and $\therefore DO=OB$, and $MB=\frac{1}{2}AB=\frac{1}{2}DC=DN$.

Hence in $\triangle$s DON, MOB,

$\because DO=BO$, and $MB=DN$, and $\angle MBO = \angle ODN$,

$\therefore \angle MOB = \angle DON$; that is, $\angle POB = \angle QOD$;

$\therefore \angle POB$ is the supplement of $\angle QOB$, and $\therefore POQ$ is a st. line.

9. Since O is the middle pt. of BD, and OF is $\parallel$ to BK,

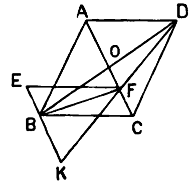


Fig. 141.

$\therefore F$ is the middle pt. of DK.

$\therefore \triangle BKF = \triangle FBD$;

$\therefore \triangle FEK$=sum of $\triangle$s BKF, FEB,

=sum of $\triangle$s FBD, FBC,

$= \triangle DBC$ diminished by $\triangle DFC$,

$= \triangle ABC$ diminished by $\triangle FBC$

(See Ex. 2 on p. 72.)

$= \triangle ABF$.

10. Let the alternate sides of the polygon $ABCDE$ be produced to meet in F, G, H, K, L.

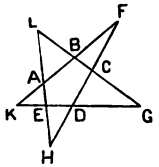


Fig. 142.

Then sum of intr. $\angle$s ABC, BCD, CDE, DEA, EAB,
$=$ sum of $\angle$s BFC, FCB, CGD, GDC, DHE, HED, EKA, KAE, ALB, LBA,
$=$ sum of $\angle$s FCB, GDC, HED, KAE, LBA, together with $\angle$s at F, G, H, K, L,
$=$ four rt. $\angle$s together with $\angle$s at F, G, H, K, L.
(I. 32, Cor. 2.)

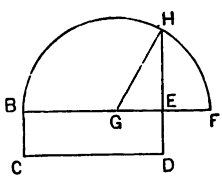


Fig. 143.

11. Take the diagram of II. 14.
Then perimeter of rect. $BEDC =$ two BF,
$=$ four BG,
$=$ four GH;
and perimeter of sq. on $HE =$ four HE;
$\therefore$ since GH is greater than HE, perimeter of rect. $BCDE$ is greater than perimeter of sq. on HE.

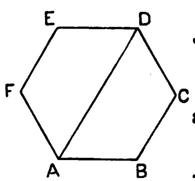


Fig. 144.

12. Let $ABCDEF$ be an equiangular hexagon.
Join DA.
Then $\angle\ ABC = \frac{4}{3}$ of a rt. $\angle$, (P. 55, Ex. 2.)
and $\angle\ BCD = \frac{4}{3}$ of a rt. $\angle$,
and sum of $\angle$s $ABC, BCD, CDA, DAB =$ four rt. $\angle$s;
(I. 32, Cor. 1.)
$\therefore$ sum of $\angle$s CDA, $DAB = \frac{4}{3}$ of a rt. $\angle = \angle\ CDE$;
$\therefore \angle\ EDA = \angle\ DAB$, and $\therefore ED$ is $\parallel$ to AB.

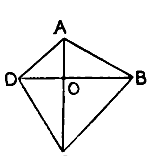


Fig. 145.

13. Let AC, BD, equal st. lines, intersect at rt. $\angle$s in O.
Complete the quadrilateral $ABCD$.
Then twice area of $\triangle\ ABC =$ rect. AC, BO,
and twice area of $\triangle\ ADC =$ rect. AC, DO;
$\therefore$ twice area of $ABDC =$ rect. AC, BD,
$=$ sq. on AC.

14. (1.) Let $AC=BD$, and $AD=BC$.

Then $\triangle$s ADC, BDC are equal in all respects.

and $\therefore$ CD is $\parallel$ to AB. (I. 39.)

(2.) Let $AC=BC$, and $AD=BD$.

Produce CD to meet AB in E.

Then, as in (1.), $\triangle$s ADC, BDC are equal in all respects,

$\therefore \angle DCA = \angle BCD$, and $\therefore$ CE is $\perp$ to AB.

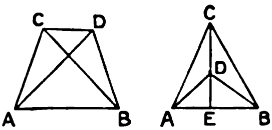


Fig. 146.

15. Bisect $\angle ABC$ by BE, meeting AC in E.

Draw $ED \parallel$ to BC, and $\therefore \perp$ to AC.

Then $\angle DEB = \angle EBC$,

$= \angle DBE$;

and $\therefore$ $DE=DB$.

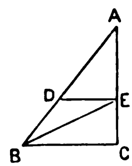


Fig. 147.

16. Let ACB, ADB be any two $\triangle$s of equal area on the base AB, and on the same side of it.

Join CD. Then, by I. 39, CD must be $\parallel$ to AB.

$\therefore$ the locus is a st. line, passing through C, D, $\parallel$ to AB.

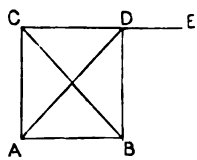


Fig. 148.

17. Let ACB be an isosceles $\triangle$, and ADB any other $\triangle$ of equal area on the same base. Join DC and produce it to E; then ECD is $\parallel$ to AB. (I. 39.)

Then $\because \angle ECA = \angle CAB$,

and $\angle BCD = \angle CBA$,

$\therefore \angle ECA = \angle BCD$;

$\therefore$, by Ex. 7, p. 116, the sum of AC, CB is less than the sum of AD, DB;

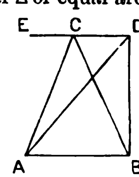


Fig. 149.

$\therefore$ *perimeter of* $\triangle$ *ACB is less than perimeter of* $\triangle$ ADB.

18. Let ABC be an isosceles $\triangle$, having $\angle\ BAC$ four times as great as either of the other $\angle$s.

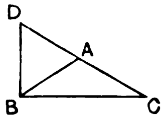


FIG. 150.

Draw $BD \perp$ to BC, meeting CA produced in D.
Then $\angle\ ABC = \frac{1}{6}$ of two rt. $\angle$s $= \frac{1}{3}$ of a rt. $\angle$;
$\therefore \angle\ ABD = \frac{2}{3}$ of a rt. $\angle = \frac{1}{3}$ of two rt. $\angle$s.
Also, $\angle\ BAC = \frac{4}{6}$ of two rt. $\angle$s,
and $\therefore \angle BAD = \frac{1}{3}$ of two rt. $\angle$s.
$\therefore \angle\ ADB =$ supplement of sum of $\angle$s BAD, ABD
$= \frac{1}{3}$ of two rt. $\angle$s,
$\therefore ABD$ is an equilateral $\triangle$.

19. Let BC, terminated by AD, DE, two of the sides of $\triangle\ ADE$, be bisected in O. Join AO, and produce it to F, so that $AO = FO$. Join BF, CF.

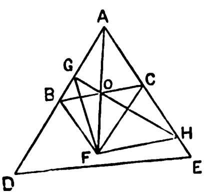


FIG. 151.

Then $ABFC$ is a $\square$. (P. 59, Ex. 2.)

Again, let GH be any other line passing through O and terminated by AD, AE, and, if it be possible, let $GO = HO$. Join GF, HF.

Then $AGFH$ is a $\square$. (P. 59, Ex. 2.)

and $\therefore CF$, HF are both $\parallel$ to AB, which is absurd.

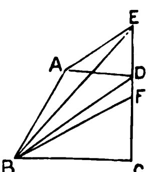


FIG. 152.

20. Let $ABCD$ be a quadrilateral. Join BD.
Draw $AE \parallel$ to BD, meeting CD produced in E.
Bisect EC in F. Join BE, BF. Then BF shall bisect the quadrilateral.
For since $\triangle\ BAD = \triangle\ BED$, add to each $\triangle\ BDF$;
then fig. $BADF = \triangle\ EFB$,
$= \triangle\ FBC$, because $FC = FE$.

21. Let AB, CD be the diagonals. Place them so as to bisect each other at rt. $\angle$s in O.

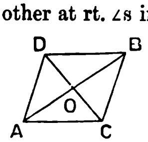


FIG. 153.

Then $ACBD$ is the rhombus reqd.
For $\because DO = CO$, and OB is common,
and $\angle\ DOB = \angle\ COB$
$\therefore DB = CB$.
Similarly it may be shown that $DB = DA$, and that $CB = CA$.
$\therefore ACBD$ is a rhombus.

22. Describe a rectangle $EFGH$ = fig. $ABCD$. (I. 45.)

Draw $AN = \frac{1}{2}$ the given altitude, and $\perp$ to AB.

To AN apply the rect. $ANOP$ = rect. $EFGH$.

Produce OP to Q, making $PQ = 2AN$, and join AQ.

Then $\triangle APQ = \frac{1}{2}$ rect. AP, PQ,
$= $ rect. $AP, AN = ANOP$
$= EFGH = ABCD$.

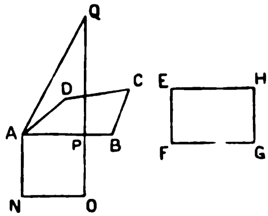


FIG. 154.

23. From O, any pt. in AB, the base of the isosceles $\triangle ACB$, draw OD, OE $\perp$ to AC, BC, and draw AF $\perp$ to BC.

Produce EO to N, meeting AN drawn $\parallel$ to FE.

Then $AFEN$ is a rectangle, and $AF = NE$.

Now $\angle OAN = \angle CBA = \angle OAD$.

Then $\because \angle OAN = \angle OAD$, and
$\angle ODA = \angle ONA$, and AO is common,
$\therefore OD = ON$.

$\therefore AF = NE$ = sum of ON, OE = sum of OD, OE.

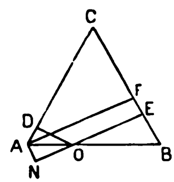


FIG. 155.

24. Sum of sqq. on AC, AD = twice rect. AC, AD with sq. on CD: (II. 7.)

$\therefore$ sqq. on AC, CB, AD = twice rect. AC, AD with sqq. on CD, CB;

$\therefore$ sqq. on AB, AD = twice rect. AC, AD, with sq. on DB. (1.)

Again, sum of sqq. on AB, AE = twice rect. AB, AE with sq. on EB: (II. 7.)

$\therefore$ sqq. on AB, AE, ED = twice rect. AB, AE, with sqq. on EB, ED;

$\therefore$ sqq. on AB, AD = twice rect. AB, AE with sq. on DB. (2.)

$\therefore$, comparing (1) and (2), we have rect. AC, AD = rect. AB, AE.

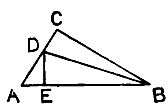


FIG. 156.

25. Let EF bisect the $\square$ $ABCD$. Join EB, EC.

Then $\triangle EBC = \frac{1}{2} \square ABCD$: (I. 41.)

$\therefore \triangle EBC$ = quadrilateral $EDCF$;
take from each $\triangle EFC$,
then $\triangle EBF = \triangle CED$.

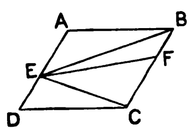


FIG. 157.

26. Draw GM, $FN \perp$ to DB, EC, produced, and draw $AO \perp$ to BC.

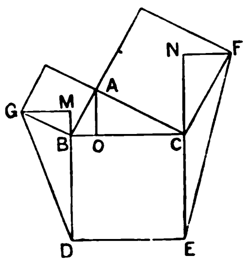


Fig. 158.

Then $\because \angle MBG = \angle ABO$, and
$\angle GMB = \angle AOB$, and $GB = AB$,
$\therefore MB = BO$, and similarly $CN = CO$.
Now sq. on GD
$=$ sqq. on DB, BG with twice rect. DB, BM,
$=$ sqq. on BC, AB with twice rect. BC, BO
And sq. on EF
$=$ sqq. on EC, CF with twice rect. EC, CN
$=$ sqq. on BC, AC with twice rect. BC, CO
$\therefore$ sqq. on GD, EF
$=$ sqq. on BC, AB, BC, AC with twice sq. on BC. (II. 2.
$=$ five times sq. on BC

27. Let ABC be any triangle.

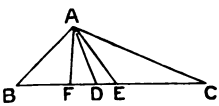


Fig. 159.

Let AD bisect $\angle BAC$, meeting BC in D,
let AF be $\perp$ to BC, meeting BC in F,
let AE bisect BC in E.
Then we can show, as in Ex. 6 on p. 41, that AE is greater than AD.
Also since $\angle AFD$ is a rt. angle, AD is greater than AF.
Again, D must lie between E and F, as is proved in Ex. 3, p. 32.

28. Since AC is bisected in E,
sum of sqq. on AB, BC = twice sum of sqq. on BE, EC: (II. 13, Ex.)
and since AB is bisected in F,

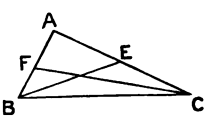


Fig. 160.

sum of sqq. on AC, CB = twice sum of sqq. on CF, FB. (II. 13, Ex.)
$\therefore$ sqq. on AB, AC, BC, BC = twice sqq. on BE, EC, CF, FB;
$\therefore$ three times sq. on BC = twice sqq. on BE, CF with twice sqq. on EC, FB;
$\therefore$ six times sq. on BC = four times sqq. on BE, CF with four times sqq. on EC, FB.
Now four times sqq. on EC, FB = sum of sqq. on AC, AB. (II. 2, Ex.)
$=$ sq on BC;
$\therefore$ five times sq. on BC = four times sqq. on BE, CF.

29. Sq. on AD = sum of sqq. on DE, AE:
sq. on DB = sum of sqq. on DF, BF.
$\therefore$ sum of sqq. on AD, DB = sum of sqq. on DE, AE, DF, BF.
So also, sum of sqq. on AC, CB = sum of sqq. on CE, AE, CF, BF.
Now sum of sqq. on AD, DB = sq. on AB
= sum of sqq. on AC, CB;
$\therefore$ sum of sqq. on DE, AE, DF, BF
= sum of sqq. on CE, AE, CF, BF.
$\therefore$ sum of sqq. on DE, DF = sum of sqq. on CE, CF.

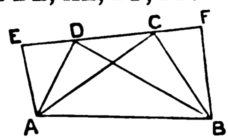


FIG. 161.

30. Since E and G are the middle pts. of AD, AB,
$\therefore GE$ is $\parallel$ to BD. (I. 40, Ex. 1.)
Since F and H are the middle pts. of DC, CB,
$\therefore FH$ is $\parallel$ to BD. (I. 40, Ex. 1.)
$\therefore GE$ is $\parallel$ to FH,
and GH is $\parallel$ to EF. (I. 40, Ex. 1.)
$\therefore GEFH$ is a $\square$, and $\therefore GE = HF$.

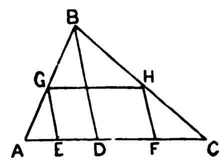


FIG. 162.

31. Draw $AE \perp$ to BC, bisecting BC in E.
Then sq. on AB = sum of sqq. on AE, EB,
and sq. on AD = sum of sqq. on AE, ED;
$\therefore$ difference of sqq. on AB, AD
= difference of sqq. on EB, ED,
= rect. BD, DC. (II. 5.)

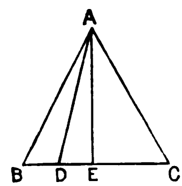


FIG. 163.

32. In AB, BC, CD, DA, the sides of a square, take E, F, G, H equidistant from A, B, C, D. Join $EFGH$.
Then $\because AE = FB$, and $AH = BE$, and $\angle EAH = \angle EBF$,
$\therefore EH = EF$, and $\angle AEH = \angle BFE$
Similarly, $EH = HG = GF = FE$.
Also, $\angle AEF$ = sum of $\angle$s EBF, BFE,
= sum of $\angle$s EBF, AEH;
and $\therefore \angle HEF = \angle EBF$ = a rt. angle.
$\therefore EFGH$ is a square.

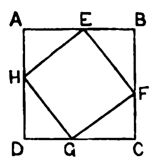


FIG. 164.

33. Let the sides of the equilateral and equiangular pentagon $ABCDE$ be produced to meet in M, N, O, P, Q.

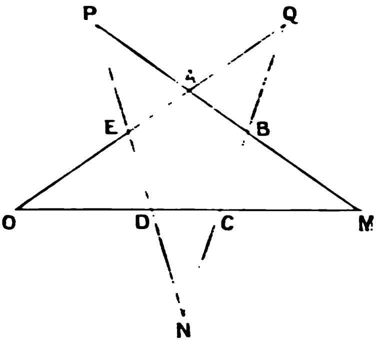


FIG. 165.

Then $\angle\, CBM$ = sum of $\angle$s at P and N,
and $\angle\, BCM$ = sum of $\angle$s at O and Q;
$\therefore$ sum of $\angle$s at P, Q, M, N, O = the three $\angle$s of $\triangle\, BCM$.
= two rt. angles.

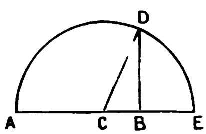


FIG. 166.

34. Let AC, CB, placed in the same st. line, be sides of the unequal squares.
Produce AB to E, making $CE = CA$.
On AE describe a semicircle ADE.
Draw $DB \perp$ to AE. Join CD.
Then sq. on BD = diff. of sqq. on CD, CB,
= diff. of sqq. on AC, CB.

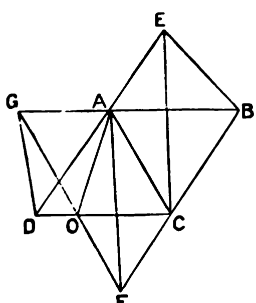


FIG. 167

35. Join EB, AO.

Then $\triangle\, AEB = \triangle\, EAC$, on same base EA,
$= \triangle\, AFC$, (I. 34.)
$= \triangle\, AOC$, on same base AC,
$= \triangle\, AOG$, (I. 34.)
$= \triangle\, AGD$, on same base AG.

36. Fig. $BADE$ = one-fourth sq. on AD,
= one-eighth sq. on AC,
= one-eighth of sixteen times sq. on AE,
= twice sq. on AE.

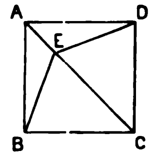


FIG. 168.

37. Bisect $\angle\ ACB$ by CD, meeting AB in D, and draw $CE \perp$ to AB.

Then $\because \angle DAC = \angle ACD$, $\therefore AD = CD$;
and $\because \angle BDC$ = sum of $\angle$s DAC, ACD,
$\therefore \angle BDC = \angle DBC$,
and $\therefore CD = BC$, and $\therefore DE = BE$.

Then sq. on AB
= sq. on AC,
= sum of sqq. on AE, EC,
= sq. on AE with difference of sqq. on BC, BE,
= sq. on BC with difference of sqq. on AE, BE,
= sq. on BC with rect. AB, AD,
= sq. on BC with rect. AB, BC.

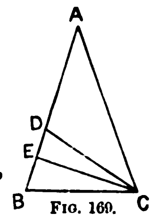


FIG. 169.

38. Let $ABCD$ be a trapezium, having $AB \parallel$ to CD.

Bisect AD in E, and join BE, CE.

Through E draw $FEG \parallel$ to BC, meeting BA, or BA produced, in F, and CD, or CD produced, in G.

Then $\because \angle EFA = \angle EGD$,
and $\angle AEF = \angle GED$, and $AE = DE$;
$\therefore \triangle EFA = \triangle EGD$,
and $\therefore \square\ BFGC$ = trapezium $ABCD$.
Hence $\triangle BEC$ = half of $\square\ BFGC$,
= half of trapezium ABCD.

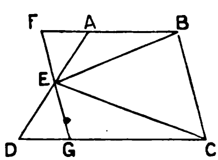


FIG. 170.

39. Let $ABCD$ be the given $\square$, and E the line of given altitude.

Draw $CF \perp$ to BC and equal to E.

Through F draw $HFG \parallel$ to AD, meeting BA and CD, or these produced, in H and G.

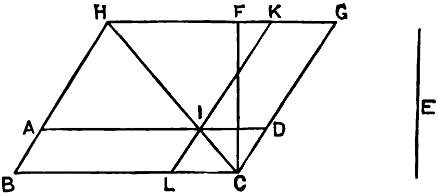


FIG. 171.

Join HC meeting AD in I, and draw $KIL \parallel$ to HB, meeting HG, BC in K and L.

Then $KLCG$ is the $\square$ reqd.

For $KD = AL$ (I. 43), and $\therefore KC = AC$.

Also KC, AC are equiangular, and the altitude of KC is $CF = E$.

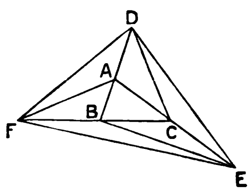


FIG. 172.

40. Let ABC be a $\triangle$. Produce BA to D, AC to E, and CB to F, each to twice its original length.

Join AF, BE, CD, DF, FE, ED.

Then $\triangle\, DEF =$ sum of $\triangle$s ABC, ABF, ADF, ADC, DCE, CBE, EBF, and each of these $\triangle$s $= \triangle\, ABC$; (I. 38, Ex. 1.)

$\therefore \triangle\, DEF =$ seven times $\triangle\, ABC$.

41. Let ABC be a right-angled $\triangle$, with $\angle\, BAC$ a rt. $\angle$, and $BA = CA$.

Let BD, the bisector of $\angle\, ABC$, meet AC in D.

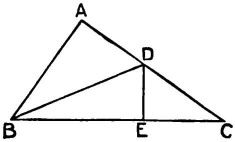


FIG. 173.

Draw $DE \perp$ to BC.

Then $BE = BA$, and $DE = DA$; (I. 26.) and $\because \angle\, ECD = \frac{1}{2}$ a rt. $\angle$, $\therefore \angle\, EDC = \frac{1}{2}$ a rt. $\angle$, and $\therefore ED = EC$.

Then sq. on $CD =$ sum of sqq. on EC, ED,

$=$ twice sq. on ED,

$=$ twice sq. on DA.

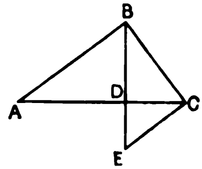


Fig. 174.

42. Rect. BD, DE = sq. on CD; (II. 4, Ex.)
Rect. AD, DC = sq. on BD; (II. 4, Ex.)
$\therefore$ rect. BD, DE with rect. AD, DC
= sum of sqq. on CD, BD,
= sq. on BC.

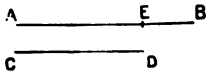


Fig. 175.

43. Let AB, CD be two st. lines, of which AB is the greater.
In AB take $AE = CD$.
Then sum of sqq. on AB, AE = twice rect. AB, AE with sq. on BE;
$\therefore$ sum of sqq. on AB, CD = twice rect. AB, CD with sq. on BE;
$\therefore$ sum of sqq. on AB, CD is greater than twice rect. AB, CD.

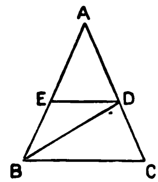


Fig. 176.

44. Let ABC be a $\triangle$, having $AB = AC$.
Bisect $\angle ABC$ by BD, meeting AC in D.
Draw $DE \parallel$ to BC, meeting AB in E.
Then $EBCD$ is the trapezium reqd.
For $\because \angle AED = \angle ABC$, and $\angle ADE = \angle ACB$;
$\therefore \angle AED = \angle ADE$, and $\therefore AE = AD$,
and $\therefore BE = CD$.
Again, $\because ED$ is $\parallel$ to BC, $\therefore \angle EDB = \angle DBC$,
$\therefore \angle EDB = \angle EBD$, and $\therefore ED = BE$;
$\therefore BE, ED, DC$ are all equal.

45. Since the sum of the squares on the diagonals is equal to the sum of the squares on the four sides (see p. 93, Ex. 2), so long as the sides are of given length the sum of the squares on the diagonals will be the same.

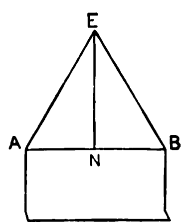


Fig. 177.

46. Let AEB be the equilateral $\triangle$. Draw $EN \perp$ to AB.
Then $BN = \frac{1}{2} AB = BC$.
Then area of rectangle = rect. AB, BN,
and area of triangle = rect. EN, BN.
Now EN is less than EB, and $\therefore EN$ is less than AB;
$\therefore$ area of triangle is less than area of rectangle.

47. Draw $BD \perp$ to CO produced.

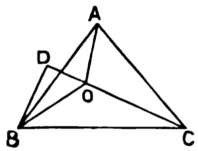


FIG. 178.

Then $\angle BOC = \frac{1}{3}$ of four rt. $\angle$s,
and $\therefore \angle BOD = \frac{1}{3}$ of two rt. $\angle$s;
and $\angle BDO$ is a rt. $\angle$,
and $\therefore BO = 2\,OD$. (P. 116, Ex. 3.)

Then sq. on BC = sum of sqq. on OB, OC with twice rect. OD, OC; (II. 12.)

$\therefore$ sq. on BC = sum of sqq. on OB, OC with rect. OB, OC.
Similarly, sq. on CA = sum of sqq. on OC, OA with rect. OC, OA;
and sq. on AB = sum of sqq. on OA, OB with rect. OA, OB;
$\therefore$ sum of sqq. on BC, CA, AB = twice sum of sqq. on OA, OB, OC with sum of the rectangles OB, OC; OC, OA; OA, OB.

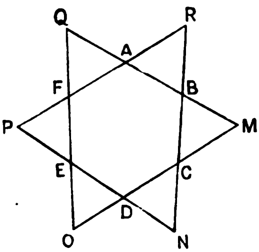


FIG. 179.

48. Let $ABCDEF$ be an equilateral and equiangular hexagon, and let the sides produced meet in M, N, O, P, Q, R.
Then MOQ is a triangle,
and $\therefore$ sum of $\angle$s at M, O, Q = two rt. $\angle$s;
and PNR is a triangle,
and $\therefore$ sum of $\angle$s at P, R, N = two rt. $\angle$s;
$\therefore$ sum of $\angle$s at M, N, O, P, Q, R
= four rt. $\angle$s.

49. Sum of sqq. on BE, ED, DC, with twice rect. BE, ED with twice rect. BE, CD, with twice rect. ED, CD,

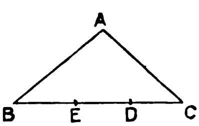


FIG. 180.

= sq. on BC, (See p. 93, Ex. 9.)
= sum of sqq. on BA, AC,
= sum of sqq. on BD, CE,
= sum of sqq. on BE, ED with twice rect. BE, ED, with sum of sqq. on ED, DC with twice rect. ED, CD.

$\therefore$, taking from each common squares and rectangles,
twice rect. BE, CD = sq. on ED.

50. Let $ABCD$ be the given square.

Produce CB to E, making BE = one side of the rectangle.

Complete the rectangle $ABEF$.

Produce FB to meet DC produced in G.

Draw $GHK \parallel$ to CE, meeting AB, FE, produced, in H, K.

Then rect. $BEKH$=square $ABCD$.

(I. 43.)

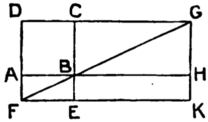


FIG. 181.

51. Sum of sqq. on AD, CD with twice rect. AD, CD,

=sq. on AC,

=sq. on AB,

=sum of sqq. on AD, BD;

$\therefore$ sq. on BD=sq. on CD with twice rect. AD, CD.

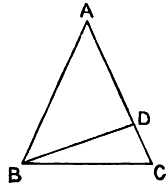


FIG. 182.

END OF BOOK II.

BOOK III.

Page 125.

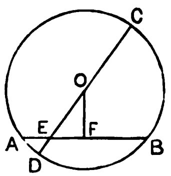


FIG. 183.

EXERCISE 1. Let CD cut AB, but not at right angles.

From O, the centre, draw $OF \perp$ to AB.

Then $AF = BF$, and $\therefore AE$ is not equal to BE.

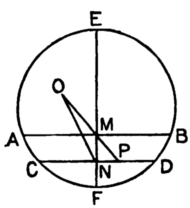


FIG. 184.

Ex. 2. Let EF bisect each of the || chords, AB, CD, in the pts. M, N.

Then the centre of the ⊙ must be in EF.

For, if not, let O be the centre, and join OM, ON, and produce OM to meet CD in P.

Then OM is $\perp$ to AB, and $\therefore OP$ is $\perp$ to CD.

But ON is also $\perp$ to CD, which is absurd.

$\therefore$ centre of ⊙ lies in EF;

$\therefore EF$ bisects AB, CD at rt. $\angle$s.

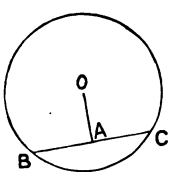


FIG. 185.

Ex. 3. Let A be the given pt.

Find O the centre of the ⊙.

Join OA; draw $AB \perp$ to OA meeting the ○ce in B. Produce BA to meet the ○ce in C.

Then since OA is $\perp$ to BC, it bisects BC.

Page 126.

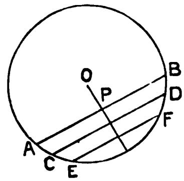


FIG. 186.

EXERCISE 1. From O the centre of the ⊙ draw a st. line OP, bisecting one of the chords, AB.

$\therefore$ OP is $\perp$ to AB;

$\therefore$ OP is $\perp$ to the other chords, CD, EF, . . . ;

$\therefore$ OP, produced if necessary, bisects all the chords;

$\therefore$ the locus is a st. line passing through the centre.

Ex. 2. Let $ABCD$ be a ▱ inscribed in the ⊙ $ABCD$, then must it be a rectangle.

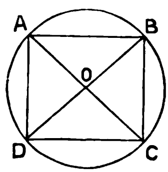


FIG. 187.

For since the diagonals of the ▱ bisect each other in O, O must be the centre of the ⊙, and $\therefore$ $CA = BD$.

Then

$\because$ $AB = DC$, and BC is common, and $CA = BD$;

$\therefore$ $\angle ABC = \angle BCD$;

$\therefore$ each of these $\angle$s is a right angle. (I. 29.)

Similarly, each of the $\angle$s BAD, ADC is a right angle.

Page 127.

EXERCISE. Let ABC, ABD be two circles cutting one another in the points A and B.

Let O be the centre of ⊙ ABC, and P the centre of ⊙ ABD.

Join OP. Through B draw $CBD \parallel$ to OP.

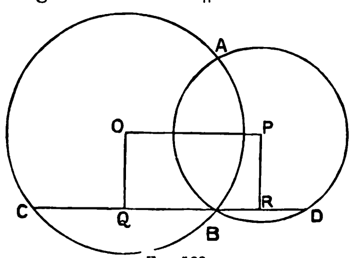


FIG. 188.

Draw OQ, $PR \perp$ to CD.

Then $OPRQ$ is a rectangle,

and $\therefore$ $QR = OP$.

Now $CD = CB + BD$

$= 2QB + 2BR = 2QR = 2OP$.

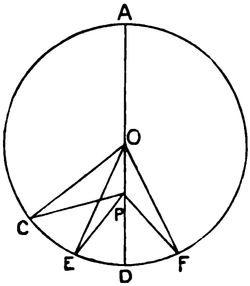


FIG. 189.

Page 130.

EXERCISE 1. Let PE be any ot drawn from P to the ○ce. Join

Then sum of OP, PE is grea OE;

$\therefore$ sum of OP, PE is greater tha

$\therefore$ sum of OP, PE is greater tha OP, PD;

$\therefore$ PE is greater than PD.

Ex. 2. Let PE, PC be two positions of PB (fig. to Ex. 1), nearer to D than C is.

Join OC, OE.

Then $\because$ $CO=EO$, and OP is common, and $\angle\ COP$ is grea $\angle\ EOP$,

$\therefore$ CP is greater than EP.

Ex. 3. Draw OF making $\angle\ POF = \angle\ POE$, and join FP Ex. 1).

Then $\because$ $OE=OF$, and OP is common, and $\angle\ POE = \angle\ PO$

$\therefore$ $PE=PF$.

But any other line drawn from P to the ○ce may be shov not equal to PE (or PF) as in Ex. 2.

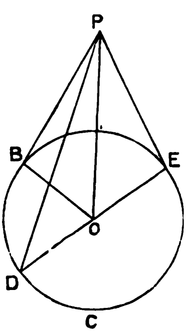


FIG. 190.

Page 131.

EXERCISE 1. Let D be any pt. in the (tween B and C.

Then shall PD be greater than PB.

Join BO, DO.

Then $\because$ $BO=DO$, and OP is comm $\angle\ DOP$ is greater than $\angle\ BOP$,

$\therefore$ PD is greater than PB.

Ex. 2. Make $\angle POE = \angle POB$, and join PE (fig. to Ex. 1).
Then $\because BO = EO$, and OP is common, and $\angle BOP = \angle EOP$,

$$\therefore PB = PE.$$

But any other line, drawn from P to meet the ○ce, may be shown not to be equal to PB (or PE) as in Ex. 1.

Page 135.

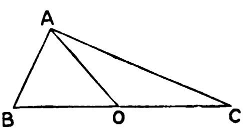


FIG. 191.

EXERCISE. Let $\angle BAC$ be a right angle.
Make $\angle BAO = \angle ABC$,
and $\therefore \angle CAO = \angle ACB$.
Then $OB = OA = OC$.
$\therefore O$ is the centre of the ⊙ described round ABC.

Page 137.

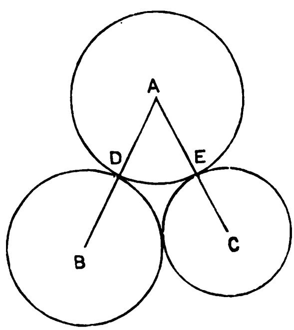


FIG. 192.

EXERCISE. Let the ⊙, whose centre is A, touch the ⊙s, whose centres are B and C, in D and E respectively.

Then difference between AB and AC,

= difference between DB and EC,

= difference between radii of ⊙s whose centres are B and C,

= half the difference between diameters of those ⊙s.

Page 140.

Take the diagram of the Proposition, and $OC = 5$ inches and $CQ = 4$ inches.

Then since $\sqrt{25-16} = \sqrt{9} = 3$, $\therefore OQ = 3$ inches;

$\therefore$ the second chord is equal to the first.

Page 142.

Exercise 1. Let AB be the given chord, C the given line. Find O the centre of the ⊙, and draw EOF a diameter.

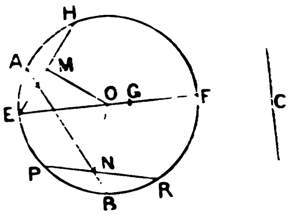


Fig. 193.

In EF take $EG=C$, and with centre E and distance EG describe a ⊙ cutting the given ⊙ in H. Join EH, and bisect it in M.

With centre O and distance OM describe a ⊙ cutting AB in N. Through N draw the chord $PR \perp$ to ON.

Then PR and EH being equidistant from the centre O are equal, and $PR=C$, and PR is bisected by AB.

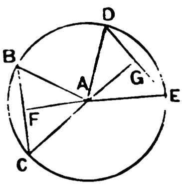


Fig. 194.

Ex. 2. Place the △s ABC, ADE so that their vertices coincide in A.

Then since AB, AC, AD, AE are all equal, a ⊙ described with centre A and distance AB will pass through C, D, E. And since the ⊥s AF, AG drawn from A to the chords BC, DE, are equal, ∴ $BC=DE$.

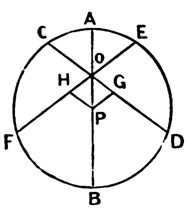


Fig. 195.

Ex. 3. Let AB be a diameter of the ⊙, and let the chords CD, EF cut AB in O, so that $\angle AOC = \angle AOE$.

From P, the centre, draw PG, PH ⊥s to CD, EF.

Then $\angle POG = \angle AOC = \angle AOE = \angle POH$.

Then ∵ $\angle POG = \angle POH$,

and $\angle OHP = \angle OGP$, and OP is common,

∴ $PG=PH$, and ∴ $CD=EF$.

Page 144.

Exercise 1. Take the diagram of the Proposition, and join OD. Then in △s ABO, ADO,

∵ $BO=DO$, and OA is common, and ∠s OBA, ODA are rt. ∠s,

∴ $AB=AD$.

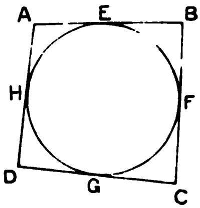


FIG. 196.

Ex. 2. Let AB, BC, CD, DA touch the ⊙ in the pts. E, F, G, H.

Then $AE = AH$, $BE = BF$, $CF = CG$, and $DG = DH$.

∴ sum of AB, CD = sum of AE, BE, CG, DG,
= sum of AH, DH, BF, CF,
= sum of AD, BC.

Page 145.

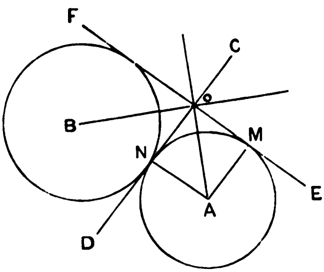


FIG. 197.

EXERCISE. Let A and B be the centres of two ⊙s that touch both the lines CD, EF, which intersect in O.

Then A must lie in the line OA that bisects ∠ DOE, since $OM = ON$, and $AM = AN$.

Similarly B must lie in the line OB bisecting ∠ FOD. And since sum of ∠s EOD, FOD = two rt. ∠s,

∴ sum of ∠s AON, BON, that is, ∠ AOB, is a right angle.

Page 146.

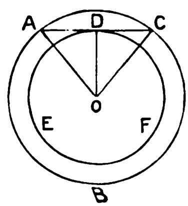


FIG. 198.

EXERCISE. Let ABC, DEF be two concentric ⊙s, and let AC, a chord of the greater, touch DEF in D.

Then shall $DC = DA$.

Find O, the common centre.

Then ∵ $OC = OA$, and OD is common, and rt. ∠ ODC = rt. ∠ ODA,

∴ $DC = DA$.

Page 148.

Exercise. Let CB touch the $\odot$ ABF in B.

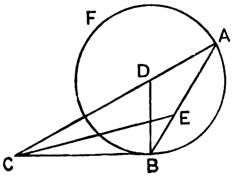


Fig. 199.

Draw CDA through D the centre.
AB, DB.
Bisect ACB by CE, meeting AB in
Then $\angle$ CEB = sum of $\angle$s ACE, DAE,
= sum of $\frac{1}{2}$ $\angle$ DCB and $\frac{1}{2}$ $\angle$
Now $\angle$s DCB, BDC together = a rt.
cause $\angle$ DBC is a rt. $\angle$;
$\therefore$ $\angle$ CEB = half a right angle.

Page 151.

Exercise 1. For taking the diagram in the Proposition and j
OB, OC in fig. 2, the reflex angle BOC is double of $\angle$ BAC, an
of $\angle$ BDC, and $\therefore$ $\angle$ BAC = $\angle$ BDC.

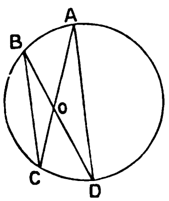


Fig. 200.

Ex. 2. Let AC, BD be chords intersecting
Join AD, BC.
Then $\because$ $\angle$ CBO = $\angle$ DAO, in same segmen
and $\angle$ BCO = $\angle$ ADO, in same segment
$\therefore$ Δs BOC, AOD are equiangular.

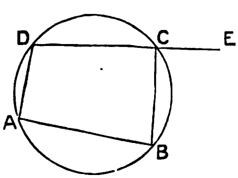


Fig. 201.

Page 153.

Exercise 1. Let $ABCD$ be a q
lateral inscribed in a $\odot$.
Produce DC to E.
Then $\angle$ ECB = supplement of $\angle$ BC
= $\angle$ BAD.

Ex. 2. Since

$\angle EBC$ = supplement of $\angle ABC$,

$= \angle ADE$;

and $\angle ECB$ = supplement of $\angle BCD$

$= \angle EAD$,

$\therefore$ $\triangle$s EBC, EAD are equiangular.

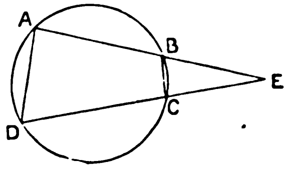


FIG. 202.

Ex. 3. From the nature of a rhombus, or any ▱ that is not rectangular, two of the opposite $\angle$s are together greater than two right angles, and therefore it cannot be inscribed in a ⊙. (See also III. 4, Ex. 2.)

Ex 4. Let $ABCD$ be a quadrilateral inscribed in a ⊙.

Produce BA to E. Bisect $\angle BCD$ by CP meeting the Oce in P. Join PA and produce it to F.

Then we have to prove that

$\angle EAF = \angle DAF$.

Now $\angle PCB = \angle PAB$ in the same segment

$= \angle EAF$;

and $\angle EAD$ = supplement of $\angle BAD$,

$= \angle BCD$;

and since $\angle PCB$ = half of $\angle BCD$,

$\therefore$ $\angle EAF$ = half of $\angle EAD$.

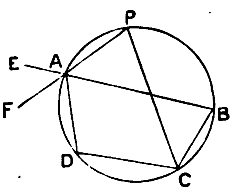


FIG. 203.

Ex. 5. Join DE.

Then $\angle ADE$ = supplement of $\angle ABC$,

= supplement of $\angle BAC$,

$= \angle BED$;

$\therefore$ $\angle CDE = \angle CED$, and $\therefore$ $CD = CE$.

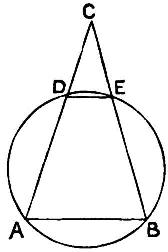


FIG. 204.

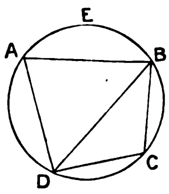


FIG. 205.

Ex. 6. Let $ABCD$ be a quadrilateral whose opposite $\angle$s are together equal to tw Then a $\odot$ $BEDC$ described about the must pass through A.

For angle in segment BED=supplei $\angle BCD$,

$= \angle BA$]

$\therefore$ A must be a pt. in the $\bigcirc$ce.

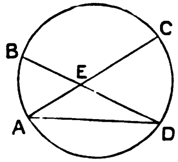


FIG. 206.

Page 156.

EXERCISE 1. Since $\angle BDA$ is of constan tude,

and $\angle CAD$ is of constant magnitude,

$\therefore$ $\angle AEB$, which is equal to the $\angle$s DBA, CAD, is of constant magnitude

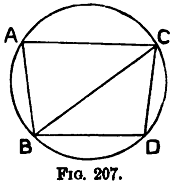


FIG. 207.

Ex. 2. Let AB, CD be equal arcs.

Join AB, BD, DC, CA, BC.

Then $\angle ACB = \angle CBD$, (

and $\therefore$ AC is $\parallel$ to BD.

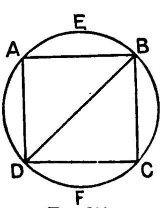


FIG. 208

Page 157.

EXERCISE 1. Join BD.

Then since chord AB=chord DC,

$\therefore$ arc AEB=arc CFD;

$\therefore \angle ADB = \angle DBC$;

$\therefore$ AD is $\parallel$ to BC.

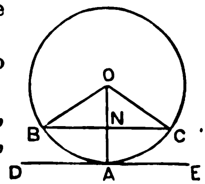


FIG. 209.

Ex. 2. Let DAE touch the ⊙ at A, the middle pt. of arc BAC.

From O, the centre, draw OA; this line is ⊥ to DAE. Let OA cut the chord BC in N.

Then ∵ arc AB = arc AC, ∴ ∠ BON = ∠ CON, and $BO = CO$, and ON is common to △s BON, CON,

∴ $BN = CN$,

and ∴ OA is ⊥ to BC;

∴ BC is ∥ to DAE.

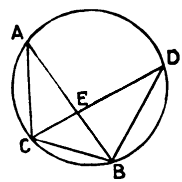


FIG. 210.

Ex. 3. Let AB, CD, equal chords, cut one another in E.

Then ∵ chord AB = chord CD,

∴ arc ACB = arc CBD;

∴ ∠ ACB = ∠ CBD.

Also, ∠ ACD = ∠ ABD, in same segment;

and ∴ ∠ ECB = ∠ EBC;

and ∴ $EB = EC$:

and ∴ $EA = ED$.

Page 158.

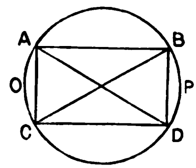


FIG. 211.

EXERCISE 1. Let AB, CD be ∥ chords in the ⊙ $ABCD$.

Join AC, BD, AD, BC.

Then ∵ ∠ ABC = ∠ BCD,

∴ arc AOC = arc BPD;

∴ chord AC = chord BD,

and ∵ ∠ ABD = supplement of ∠ BDC;

= ∠ BAC,

∴ arc ACD = arc BDC;

and ∴ chord AD = chord BC.

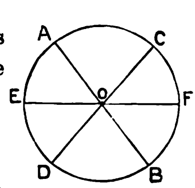


FIG. 212.

Ex. 2. Let AB, CD, EF be three equal chords in the ⊙ $ACBD$, cutting one another in the same pt. O.

Then, by III. 28, Ex. 3,

$OA = OC$, and $OA = OE$;

and ∴ *O is the centre.* (III. 9.)

Page 160.

EXERCISE. Let $CABD$ be a semicircle on the diameter CD.

From O, any pt. in CD, draw $OA \perp$ to CD, and OB to the bisection of the ○ce.

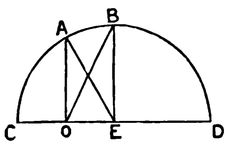


FIG. 213.

Bisect CD in E, and join BE, then BE is $\perp$ to CD. Join AE.

Then sq. on OB=sum of sqq. on OE, EB;
and sum of sqq. on OA, OE=sq. on AE;
∴ sum of sqq. on OB, OA, OE=sum of sqq. on OE, EB, AE;
∴ sum of sqq. on OB, OA=sum of sqq. on EB, AE,
=twice sq. on radius.

Page 162.

EXERCISE 1. Let AC be the diameter of the larger ⊙; and AB the diameter of the smaller ⊙

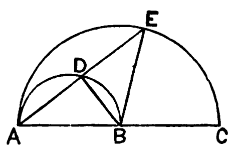


FIG. 214.

Draw ADE cutting the smaller ⊙ in D, and meeting the larger ⊙ in E.

Join DB, EB.

Then in △s ABD, EBD.

∵ ∠ ADB = ∠ EDB, and ∠ BAD = ∠ BED, and BD is common,
∴ AD=ED.

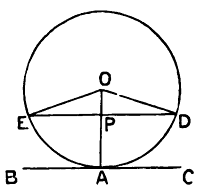


FIG. 215.

Ex. 2. Let BAC touch the ⊙ AED in A.

Let chord ED be ‖ to BAC.

From O, the centre, draw OA cutting the chord in P.

Then, since OA is $\perp$ to BAC,
∴ OP is $\perp$ to ED;
∴ PE=PD;
∴ ∠ EOP = ∠ DOP;
∴ arc EA=arc DA.

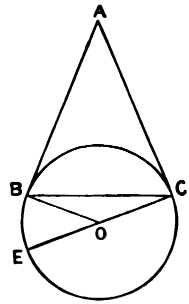


FIG. 216.

Ex. 3. Let AB, AC touch the ⊙ BCE in the pts. B, C. Join BC and through O, the centre, draw the diameter COE.

Now in the quadrilateral $ABOC$, since ∠s ABO, ACO are rt. ∠s,

∴ ∠ BOC is the supplement of ∠ BAC.

But ∠ BOC is the supplement of sum of ∠s OBC, OCB;

∴ ∠ BAC = sum of ∠s OBC, OCB;

= twice ∠ OCB.

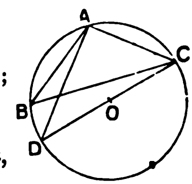


FIG. 217.

Ex. 4. Let ABC be an oblique-angled △, inscribed in a ⊙ ABC.

And let ∠ BAC be greater than a rt. ∠.

Draw COD a diameter: then ∠ DAC is a rt. ∠;

and ∠ BAD = ∠ BCD, in the same segment,

∴ ∠ BAC is greater than a rt. ∠ by ∠ BCD.

Again, ∠ ABC = ∠ ADC, in the same segment,

and ∠ ADC with ∠ ACD = a rt. ∠.

∴ ∠ ABC is less than a rt. ∠ by ∠ ACD.

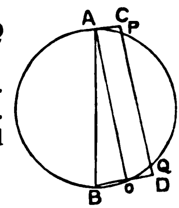


FIG. 218.

Ex. 5. Let AC, BD be the ⊥s on the chord PQ from the extremities of the diameter AOB.

Let BD, the greater perpendicular, cut the ⊙ in O.

Then ∠ AOB, being the ∠ in a semicircle, is a rt. ∠.

∴ ∠ AOD is a rt. ∠, and ∴, since the ∠s at C and D are rt. ∠s, ∠ CAO is a rt. ∠.

∴ $ACDO$ is a rectangle, and ∴ AC = OD.

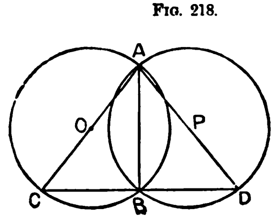


FIG. 219.

Ex. 6. Let the ⊙s ABC, ABD intersect in A and B.

Let O and P be the centres, and AOC, APD diameters.

Then ∠ ABC is a rt. ∠. (III. 31.)

Similarly, ∠ ABD is a rt. ∠;

∴ *CBD is a st. line.* (I. 14.)

Ex. 7. Let A be the common centre of the ⊙s.

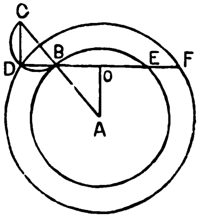


FIG. 220.

Draw AB, a radius of the smaller ⊙, an produce it to C so that $BC=AB$.

On BC describe a semicircle BDC cutting t greater ⊙ in D. Join DB and produce it to me the original circles again in E and F.

Draw $AO \perp$ to DF, and join CD.

Then ∵ rt. ∠ BDC = rt. ∠ AOB,

and ∠ ABO = ∠ CBD, and $AB=BC$,

∴ $DB=BO$, and ∴ DO= twice OB.

Now $FO=DO$, and $EO=OB$:

∴ FD= twice BE.

Page 163.

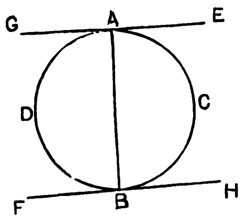


FIG. 221.

EXERCISE. Let GAE, FBH be ‖ tange to the ⊙ $ACBD$, at the pts. A, B. J AB.

Then

∠ in segment ADB = ∠ EAB,

= ∠ ABF,

= ∠ in segment AC

∴ the angles in these segments are rt. ∠

∴ AB is a diameter of the ⊙.

Page 164.

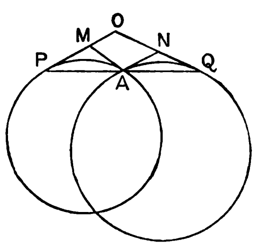


FIG. 222.

EXERCISE 1. Let the tangents at P an meet in O, and let the tangents at A m OP, OQ, in M, N.

Then ∠ POQ

= supplement of sum of ∠s MPA, NQA

= supplement of sum of ∠ s MAP, NAQ

= ∠ MAN.

Ex. 2. Let A, B be the given pts., and CD the given line. Join AB and on it describe a segment of a ⊙ capable of containing the given ∠. The points, where this segment cuts, or the point where it touches, CD, will be points, such that if lines be drawn from them to A and B, the angle contained by these lines will be equal to given ∠.

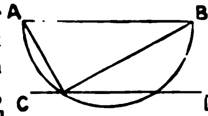


FIG. 223.

Page 165.

EXERCISE 1. Let the ⊙s ABC, ADE touch internally in A.

Draw FAE, cutting smaller ⊙ in C.

Draw the tangent PAQ, touching ⊙s at A.

Then ∠ FAQ = ∠ in segment ABC;

and ∠ FAQ = ∠ in segment ADE:

∴ segments ABC, ADE, contain equal angles.

Similarly segments AGC, AHE, contain equal angles.

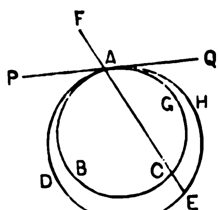


FIG. 224.

Ex. 2. Describe a ⊙ ABC with given radius. Draw AB cutting off a segment ACB capable of containing an ∠ equal to the given vertical angle, and with centre B and radius = given side, describe a ⊙ cutting ⊙ ABC in D. Join DA, DB.

Then △ DAB is the triangle required.

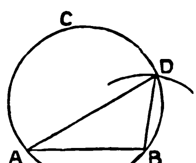


FIG. 225.

Ex. 3. Let AB be the given base. Describe on AB the segment of a ⊙ capable of containing the given vertical angle.

With centre A, and radius = the given ⊥, describe a ⊙, and draw BD touching this ⊙ in D.

Produce BD to meet the segment on AB in C.

Then ACB is the triangle required.

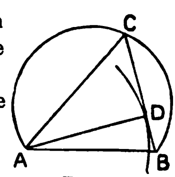


FIG. 226.

Page 166.

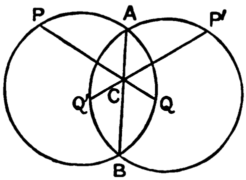


FIG. 227.

EXERCISE 1. Since rect. CP, CQ = rec
AC, CB;

and rect. CP', CQ' = rect. AC, CB

$\therefore$ rect. CP, CQ = rect. CP', CQ'.

Ex. 2. Taking diagram of Ex. 1, a ⊙ described about △ $PQ'Q$ pass through P', because rect. CP, CQ = rect. CP', CQ'.

Page 169.

Miscellaneous Exercises on Book III.

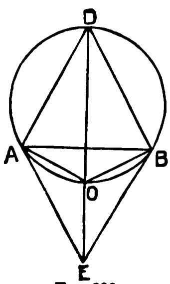


FIG. 228.

1. Let ADB, AOB be the segments. D AE, BE tangents to the ⊙. Join DE, cutt the ⊙ in O.

Then $\angle DOB$ = sum of $\angle$s OBE, OEB,
= sum of $\angle$s BDO, OEB;
and $\angle DOA$ = sum of $\angle$s ADO, OEA;
$\therefore \angle BOA$ = sum of $\angle$s ADB, AEB
$\therefore$ difference of $\angle$s BOA, ADB = $\angle AEB$.

2. Let O, P be the centres of any two of the ⊙s, A the point o contact. Then the line joining O, P passes through A. (III. 12.) Draw BC a chord of both ⊙s passing through A. Join OB, CP.

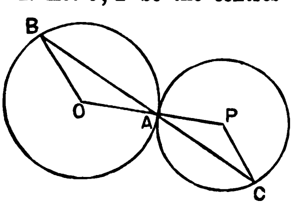


FIG. 229.

Now $\angle OBA = \angle OAB$ (I. 5.)
$= \angle PAC$ (I. 15.)
$= \angle PCA$. (I. 5.)
$\therefore$ OB is ∥ to CP.

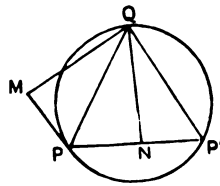


FIG. 230.

3. $\angle QPM = \angle QP'N$ in alternate segment, and $\angle QMP = \angle QNP'$, each being a rt. $\angle$;
$\therefore$ $\triangle$s QPM, $QP'N$ are equiangular.

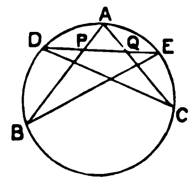


FIG. 231.

4. Let DE cut AB in P, and AC in Q.
Then $\angle AQP$ = sum of $\angle$s ACD, EDC,
= sum of $\angle$s DEB, ABE,
= $\angle APQ$;
$\therefore AP = AQ$.

5. Let $ABCD$ be the quadrilateral fig. inscribed in the $\odot$ $ABCD$, and let $\angle ABC$ be a rt. $\angle$, then $\angle ADC$ is also a rt. $\angle$. (III. 22.)

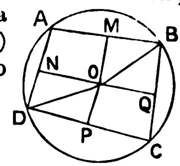


FIG. 232.

From O, the centre, draw OM, ON, OP, OQ to the middle pts. of the sides.
Then MQ, NP are rectangles.
Then sq. on OB = sum of sqq. on OM, OQ,
and sq. on OD = sum of sqq. on ON, OP;
$\therefore$ twice sq. on radius = sum of sqq. on the four perpendiculars.

6. I. Let O and A be on the same side of the centre C.
Let the chord BD be bisected in O, and let EF be any other chord through O.
Draw $CP \perp$ to EF.
Then $\because \angle CPO$ is greater than $\angle COP$,
$\therefore CO$ is greater than CP;
$\therefore FE$ is greater than BD. (III. 15.)
$\therefore$ arc EAF is greater than arc BAD;
$\therefore \angle BAD$ is greater than $\angle EAF$.

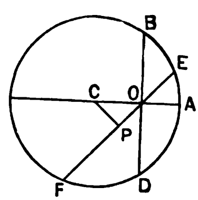


FIG. 233.

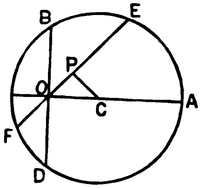


Fig. 233 *a*.

II. Let O and A be on opposite sides (
centre.

Make the same construction.

Then EF is greater than BD :

$\therefore$ arc EBF is greater than arc BFD

$\therefore$ arc EAF is less than arc BAD;

$\therefore \angle BAD$ is less than $\angle EAF$.

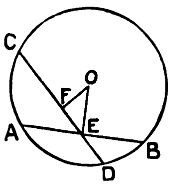


Fig. 234.

7. Let O be the centre of the $\odot$ ABC.

Bisect the chord AB in E, and draw the
CED.

Bisect CD in F and join OF.

Then $\angle OFE$ is a rt. $\angle$;

$\therefore \angle OFE$ is greater than $\angle OEF$;

$\therefore OE$ is greater than OF,

$\therefore CD$ is nearer the centre than AB is.

Similarly, if a third chord be drawn through F, it may be that this chord is nearer to the centre than CD is, and so on.

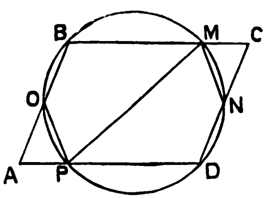


Fig. 235.

8. Let a $\odot$ pass through B,
opposite $\angle$s of the $\square$ $ABCD$. Dra
OP joining the pts. of intersection
$\odot$ and the sides of the $\square$. Join
Then

$\angle MPO$ = supplement of $\angle MBO$ (I]

= supplement of $\angle NDP$ (I.

= $\angle PMN$ (II

$\therefore MN$ is $\parallel$ to OP.

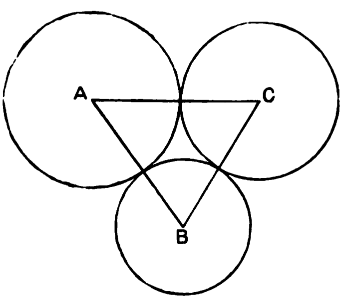


Fig. 236.

9. Let r_1, r_2, r_3 be the r
the $\odot$s, of which the cent
A, B, C.

Then AB = sum of r_1 anc
and AC = sum of r_1 and
$\therefore$ difference of AB, AC =
ence of r_2 and r_3, which is
pendent of r_1, and is th
invariable, when the circles
centres are B and C are gi

10. Take any ⊙ $ACBD$, and let O be its centre.

Draw the diameters AOB, COD at rt. ∠s.

Through A, B, C, D draw lines ∥ to CD, AB, meeting in M, N, P, Q.

Then QM, MN, NP, PQ are tangents to the ⊙ $ACBD$, and are all equal.

Then with centre O, and distance OP describe a ⊙, which will pass through P, Q M, N, and will be the outer circle reqd.

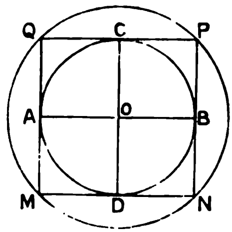


FIG. 237.

11. Let $ABCD$ be any quadrilateral inscribed in a ⊙.

Bisect any two adjacent sides, AD, AB in E, F.

Let O be the centre of the ⊙.

Then OE, OF are ⊥s on AD, AB.

∴ since ∠s AEO, AFO are rt. ∠s, a ⊙ described about △ AFE will pass through O, and AO will be its diameter.

∴ the radius of the ⊙ about AFE will be half AO; and since AO passes through the centre of this ⊙, this ⊙ will touch the ⊙ $ABCD$.

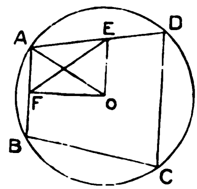


FIG. 238.

Similarly it may be shown that if any other adjacent sides be bisected, and the pts. of bisection joined, the ⊙ described about the triangle thus formed will have a radius half AO, and will touch ⊙ $ABCD$.

12. Let AB be the given st. line, MDN the given ⊙.

From O, the centre, draw OC ⊥ to AB, cutting the ⊙ in D.

Through D draw EDF ∥ to AB.

Then ∵ ∠ EDO = ∠ ACO = a rt. ∠ ;

∴ EDF is a tangent to the ⊙.

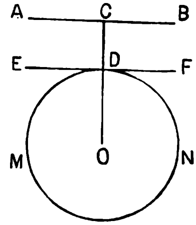


FIG. 239.

13. Let A be the given radius, DE the line in which the centre of the $\odot$ is to be, and BC the line which is to be a tangent to the $\odot$.

Draw $BF \perp$ to BC, and equal to A.

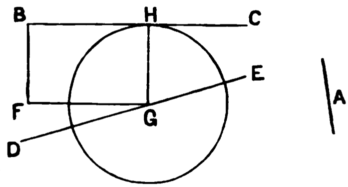


Fig. 240.

Draw $FG \parallel$ to BC, meeting DE in G.

Draw $GH \perp$ to BC.

Then a $\odot$ described with centre G and distance GH will touch BC, and will have its radius equal to A.

14. Let AB be the diameter of the given $\odot$, and O the centre. Let CD be the given line. Draw $DE = OB$, and $\perp$ to CD. Join CE. Produce AB to F, so that $OF = CE$. Draw FP a tangent to the $\odot$ at P.

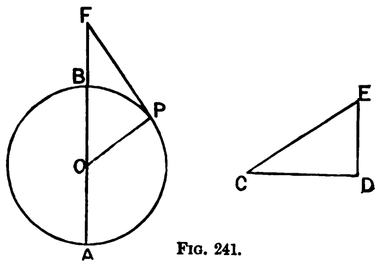


Fig. 241.

Then sum of sqq. on FP, OP = sq. on FO,

= sq. on CE,

= sum of sqq. on CD, ED;

$\therefore$ sq. on FP = sq. on CD;

$\therefore$ FP = the given line CD.

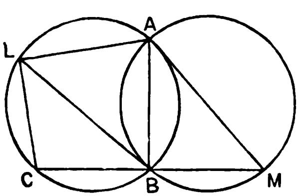


Fig. 242.

15. $\angle ABC$ = sum of $\angle$s BAM, AMB.

Now $\because LC = BM$,

$\therefore \angle LBC = \angle BAM$; (III. 28, 27.)

$\therefore \angle ABL = \angle AMB$;

$\therefore$ arc AL = arc AB. (III. 26).

Also, since $\angle LBC = \angle BAM$,

$\therefore LB$ is a tangent at B.

16. Since $\angle\, DAC = \angle\, DEA$, (III. 32.)
$= \angle\, EAB$; (I. 29.)
and $\angle\, CDA = \angle\, EBA$; (III. 32.)
$\therefore \Delta$s ACD, EAB are equiangular.

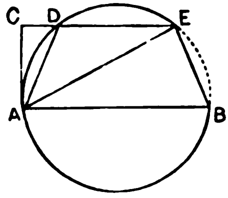


FIG. 243.

17. Since $BC = AB$,
$\therefore \angle\, CAB = \angle\, ACB$,
$= \angle\, ADB$, (III. 28, 27.)
$= \angle$ in segment ADB;
$\therefore AC$ touches the $\odot ABD$.

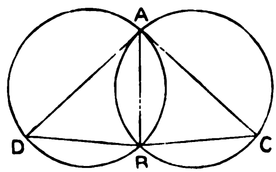


FIG. 244.

18. Let CD be the given line. Join AB. In AB take a pt. E, such that rectangle BA, AE=sq. on CD. (See p. 120, Ex. 50.) With centre B and distance BE describe a $\odot EFG$, and draw AF a tangent to this $\odot$ at F.

Then $\because$ rect. BA, AE = sq. on AF; (III. 37).
$\therefore AF = CD$.

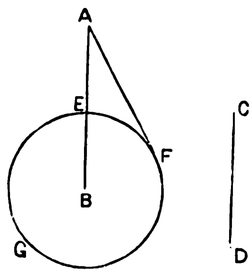


FIG. 245.

19. Let ABC be a Δ. Draw BE, CF $\perp$ to AC, AB, and let them intersect in O. Join AO and produce it to meet BC in D; then shall AD be $\perp$ to BC.

Join EF. Then a $\odot$ may be described about $AEOF$; (III. 22, Ex. 6.)
$\therefore \angle\, FAO = \angle\, FEO$. (III. 21.)
Also, a $\odot$ may be described about $BFEC$; (III. 31.)
$\therefore \angle\, FCB = \angle\, FEO$.
Hence $\angle\, FAO = \angle\, OCD$, and $\angle\, AOF = \angle\, COD$,
$\therefore \angle\, ODC = \angle\, OFA$ = a rt. $\angle$.

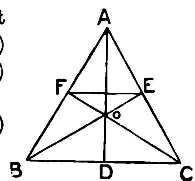


FIG. 246.

20. Let O be the intersection of the $\perp$ s as in Ex. 19.

Then, by II. 7,

sum of sqq. on AD, AO = twice rect. AD, AO with sq. on DO,
sum of sqq. on BE, BO = twice rect. BE, BO with sq. on EO,
sum of sqq. on CF, CO = twice rect. CF, CO with sq. on FO;

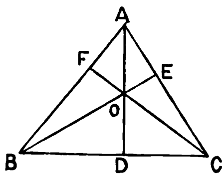


FIG. 247.

$\therefore$ sum of sqq. on AD, BO; BE, CO; CF, AO = twice sum of rects. AD, AO; BE, BO; CF, CO; with sum of sqq. on DO, EO, FO;

$\therefore$ sum of sqq. on AD, BD, DO; BE, CE, EO; CF, AF, FO = twice sum of rects. AD, AO; BE, BO; CF, CO; with sum of sqq. on DO, EO, FO;

$\therefore$ sum of sqq. on AD, BD; BE, CE; CF, AF = twice sum of rects. AD, AO; BE, BO; CF, CO;

$\therefore$ sum of sqq. on AB, BC, CA = twice sum of rects. AD, AO; BE, BO; CF, CO.

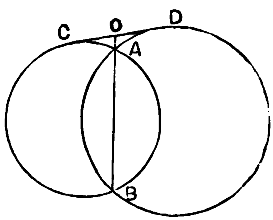


FIG. 248.

21. Let $\odot$s ABC, ABD intersect in A and B.

Draw CD a common tangent to the $\odot$s in C, D.

Let AB meet CD in O.

Then sq. on OC = rect. BO, OA, (III. 36.)
= sq. on OD;
$\therefore$ $OC = OD$.

22. $\angle\, CAG = \angle\, CBD$, in same segment;

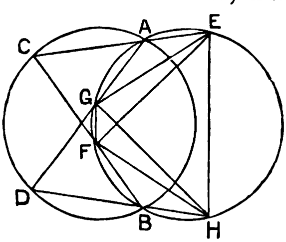


FIG. 249.

$\therefore$ $\angle\, GAE = \angle\, FBH$;
$\therefore$ $\angle\, GFE$ = supplement of $\angle\, GAE$, (III. 22.)
= supplement of $\angle\, FBH$,
= $\angle\, HGF$;
and $\angle\, GEF = \angle\, GHF$ in same segment;
$\therefore$ $\angle\, EGF = \angle\, GFH$;
$\therefore$ sum of $\angle$s GFH, FHE = two rt. $\angle$s; (III. 22.)
$\therefore$ GF is $\parallel$ to EH.

23. Since rect. $AC, CB=$ sq. on CP, (III. 37.)

and rect. $AC, CB=$ sq. on CP',

$\therefore$ sq. on $CP=$ sq. on CP';

$\therefore CP=CP'$.

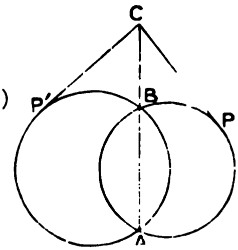


FIG. 250.

24. Let ABC be a $\triangle$, AD, BE, CF the $\perp$s from the angular pts. on the opposite sides, intersecting in O.

Then $\because$ a $\odot$ described on BC as diameter will pass through F and E, (III. 31.)

$\therefore$ rect. $CO, OF=$ rect. BO, OE (III. 36.)

Again, $\because$ a $\odot$ described on AB as diameter will pass through E and D,

$\therefore$ rect. $BO, OE=$ rect. AO, OD (III. 36.)

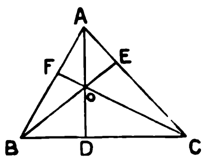


FIG. 251.

25. Let AB, AC be equal st. lines drawn from A to the $\odot$ BCD, and let them be produced to meet the $\bigcirc$ce again in D, E.

Draw OM, $ON \perp$ to BD, CE, and join OA, OB, OC.

Then $\angle ABO = \angle ACO$; (I. 8.)

$\therefore \angle OBM = \angle OCN$, and $\angle OMB = \angle ONC$,

and $OB=OC$;

$\therefore OM=ON$, and $\therefore AD$, AE are equidistant from the centre.

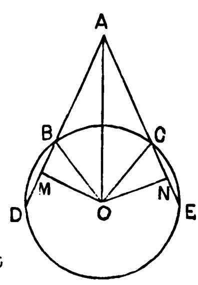


FIG. 252.

26. $\angle CBC' = \angle CBB' + \angle B'BC' = \angle CB'B + \angle BB'C' = \angle CB'C'$,

and $\angle BCB' = \text{two } \angle BAB'$,

and four right $\angle$s $- \angle BC'B' = \text{two } \angle BA'B'$;

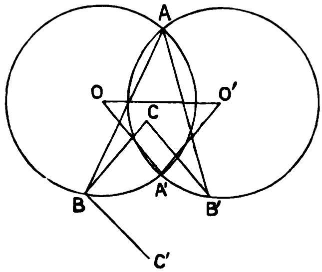


FIG. 253.

$\therefore \angle BCB' + \angle BC'B' = \text{four rt. } \angle\text{s} + \text{two } \angle BAB' - \text{two } \angle BA'B'$,

$= \text{four rt. } \angle\text{s} - \text{two } \angle ABA' - \text{two } \angle AB'A'$,

$= \text{four rt. } \angle\text{s} - \angle AOA' - \angle AO'A'$,

$= \text{two } \angle OA'O'$;

$\therefore$ two $\angle CBC' = \text{four rt. } \angle\text{s} - \text{two } \angle OA'O'$;

$\therefore \angle CBC' = \text{two rt. } \angle\text{s} - OA'O'$.

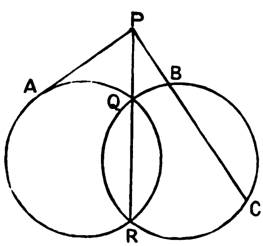


FIG. 254.

27. Let RQ the common chord of the $\odot$s AQR, BQR be produced to P. Draw PBC a chord of $\odot$ BQR.

Then rect. CP, PB = rect. RP, PQ,

= sq. on PA:

$\therefore$ a $\odot$ passing through A, B, C has PA for a tangent;

$\therefore$, PA being the common tangent to this $\odot$ and to $\odot$ AQR, these $\odot$s touch at A.

28. Let AB be the given base, and bisect it in C.

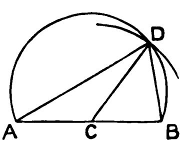


FIG. 255.

On AB describe a segment of a $\odot$ ADB capable of containing an angle = given angle.

With centre C and radius = given length of the line from the vertex to the middle pt. of the base describe a $\odot$ cutting ADB in D.

Join AD, DB. Then ADB is the $\triangle$ reqd.

29. $\angle DCB = \angle BAD$, in same segment,

$= \text{half} \angle BAC.$

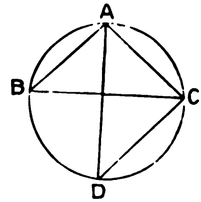


FIG. 256.

30. Let O be the intersection of AD, BC.
Then $\because \angle CBD = \angle CAD$,
and $\angle BOD = \angle AOC$,
$\therefore \angle ADB = \angle ACB$; (I. 32.)
$\therefore$ a $\odot$ passing through A, B, C will pass through D. (III. 21.)

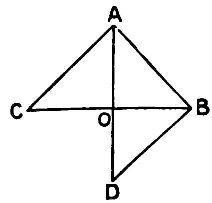


FIG. 257.

31. Obviously $AQ = AQ'$:
$\therefore \angle APQ' = \angle ABQ$; (III. 28, 27.)
$\therefore$ in $\triangle$s APT, ABR,
$\angle ARB = \angle ATP$,
$=$ a right angle.
Hence a $\odot$ may be described round $ATQR$;
and $\therefore \angle RTQ = \angle PAQ =$ complement of $\angle APB$ (since $\angle ASP$ is a right angle, for reasons similar to those by which we proved $\angle ARB$ to be a right angle).

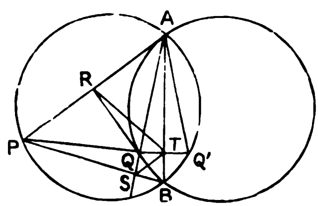


FIG. 258.

Similarly $\angle STQ =$ complement of $\angle APB$:
$\therefore \angle RTQ = \angle STQ.$

32. Let $ABCD$ be a quadrilateral such that, when DA is produced to E, $\angle EAB = \angle BCD$.
Then $\angle BAD$ is the supplement of $\angle BCD$.
$\therefore$ a $\odot$ may be described about $ABCD$.
$\therefore \angle BDA = \angle BCA$ in same segment.
Similarly it may be shown that the angles subtended by the other sides are equal.

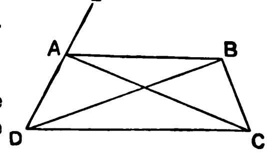


FIG. 259.

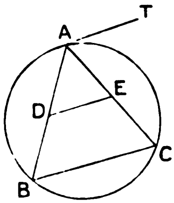


FIG. 260.

33. Describe a ⊙ about the △ ABC.

Draw AT a tangent to this ⊙.

Then ∠ TAC = ∠ ABC, (III

= ∠ ADE: (I. !

∵ AT is a tangent to the ⊙ described about ∠

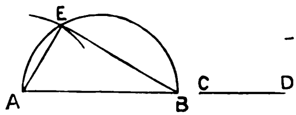


FIG. 261.

34. Let AB and CD be the of the given squares, AB bein greater. On AB describe a semi and with centre A and distanc describe a circle cutting the semi in E.

Join AE, BE.

Then sq. on EB = difference of sqq. on AB, AE,
= difference of sqq. on AB, CD.

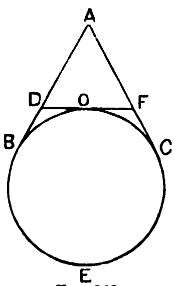


FIG. 262.

35. Let AB, AC be drawn as tangents to t BCE from the same pt. A. Draw DOF a ta to the ⊙ at O, meeting AB, AC in D, F.

Then ∵ $DO = DB$, (III. 17, E

and $FO = FC$ (III. 17, E

∴ perimeter of △ ADF = sum of AD, DO, OF
= sum of AD, DB, FC
= sum of AB, AC,

which is therefore the same for all positions o

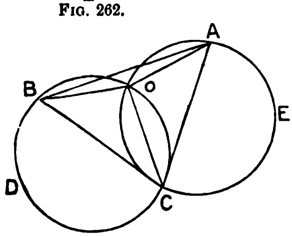


FIG. 263.

36. Let O be the pt. where the c described on AC, BC intersect.

Join AO, BO, CO.

Then ∠ AOC = supplement of ∠ segment .

and ∠ BOC = supplement of ∠ segment .

Then ∠ AOB will be supplement in segment of ⊙ described on AB

since sum of ∠ s AOB, BOC, AOC = four rt. ∠ s.

∴ the ⊙ described on AB will pass through O.

37. Produce BA to D.

Then

$\angle DAA' = \angle ABC$, (I. 29.)

$=$ supplement of sum of $\angle$ s BAC, ACB,

$=$ supplement of twice $\angle ACB$,

$=$ supplement of twice $\angle A'AE$, (I. 29.)

Now $\angle AA'E = \angle A'BC$, (I. 29.)

$= \angle A'AE$, (III. 21.)

$\therefore \angle AEA' =$ supplement of twice $\angle A'AE$;

$\therefore \angle DAA' = \angle AEA'$;

and $\therefore$ BA is a tangent to $\odot$ described about the $\triangle AEA'$.

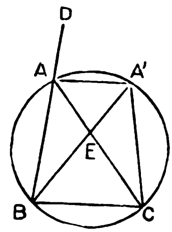


FIG. 264.

38. Let $ABCD$ be the given square.

Draw the diagonals intersecting in O.

Draw EOG, HOF $\parallel$ to AD and AB.

Bisect $\angle$s EOB, BOF, FOC, etc., by OI, OK, OL, etc., as in the diagram.

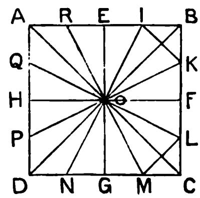


FIG. 265.

Then in $\triangle$s OBI, OBK,

$\because \angle IOB = \angle KOB$, and $\angle IBO = \angle KBO$,

and OB is common,

$\therefore OI = OK$.

Also in $\triangle$s OER, OEI,

$\because \angle EOR = \angle IOE$, and $\angle OER = \angle OEI$,

and OE is common,

$\therefore OR = OI$.

Hence OI, OK, OL, etc., are all equal.

Also, since $\angle IOK = \angle KOL$,

$\therefore IK = KL$, and similarly $KL = LM$, etc.

Therefore a $\odot$ described with centre O and distance OI will be the $\odot$ reqd., for the arcs subtended by IK, KL, LM, etc., will all be equal.

39. Let FG produced meet AB in M.

Bisect AB in C, and join CD. Bisect FG in O, and join OD, OE.

Then $\because \angle$s ADB and AEB are rt. $\angle$s, (III. 31.)

$\therefore \angle$s FDG and FEG are rt. $\angle$s;

and $\therefore$ a circle can be described about $DFEG$;

and FG is a diameter of this $\odot$, and O is its centre.

$\therefore OD = OE$.

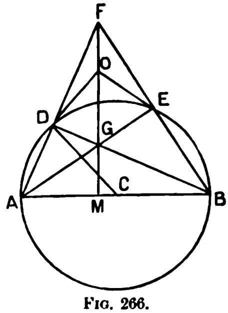


Fig. 266.

Again, since AE is ⊥ to FB, and BD is ⊥ to AF, ∴ FM is ⊥ to AB. (Ex. 19, p. 170.)

Then sum of ∠s MFA, DAM = a rt. ∠ ;

and ∠ ODF = ∠ OFD = ∠ MFA ;

∴ sum of ∠s ODF, DAM = a rt. ∠ ;

∴ ∠ ODF = ∠ DBA = ∠ CDB ;

∴ ∠ ODC is a rt. ∠, and ∴ OD is a tangent to the ⊙ ADB.

Similarly OE is a tangent to the ⊙ ADB.

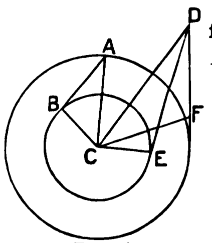


Fig. 267.

40. Let C be the common centre, AB a tangent from any pt. in outer Oce to the inner circle, DE, DF tangents to the ⊙s from any pt. D.

Join BC, AC, DC, FC, EC.

Then sum of sqq. on FC, DF = sq. on DC,
= sum of sqq. on CE, ED :
∴ difference of sqq. on DE, DF
= difference of sqq. on FC, CE,
= difference of sqq. on AC, CB,
= sq. on AB.

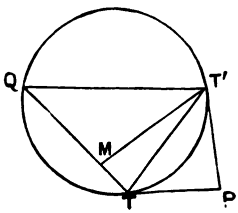


Fig. 268.

41. Sum of ∠s $QT'M$, $T'QM$ = a rt. ∠.

Now ∠ $T'QM$ = ∠ $T'TP$ = half a rt. ∠ ;

∴ ∠ $QT'M$ = half a rt. ∠ ;
= ∠ $T'QM$,

∴ $T'M = QM$.

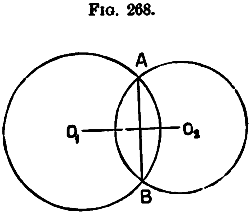


Fig. 269.

42. (1.) Let A and B be the given pts.

Then AB is a common chord of all the circles.

Join O_1, O_2 the centres of any two of the ⊙s.

Then O_1, O_2 bisects AB at rt. angles.

∴ the locus is a straight line bisecting AB at right angles.

(2.) Draw a diameter AB through the middle pt. of CD any one of the chords, this will pass through the middle pt. of each of the other chords, and will be the locus required.

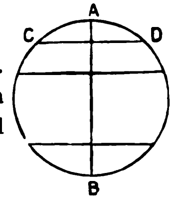


FIG. 270.

(3.) Let AB, AC be the two given lines.

Draw $AP \perp$ to AC and equal to the given st. line. Draw $PDE \parallel$ to AC, meeting BA produced in D.

Then the bisector of $\angle\ BDE$ is clearly the locus required.

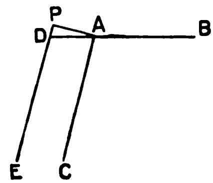


FIG. 271.

(4.) A line drawn $\perp$ to the given line AB, and passing through C the point of contact, will pass through the centre of each circle, and will be the locus required.

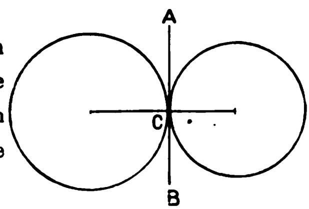


FIG. 272.

(5.) Let O be the pt. through which all the chords pass. From C the centre draw CP, $CQ \perp$ to any two of the chords. Join CO.

Then a $\odot$ described on CO as diameter will pass through P and Q, because the $\angle$s CPO, CQO are rt. $\angle$s.

$\therefore$ this $\odot$ is the locus required.

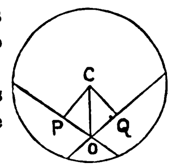


FIG. 273.

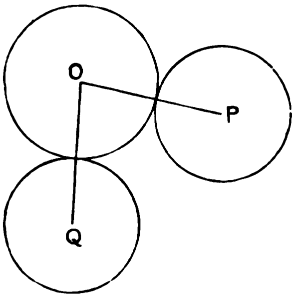


FIG. 274.

(6.) Let O be the centre given ⊙.

Then the centres (as P, Q) of ⊙s of given radius are equi from O, and ∴ the locus is a ⊙ centre is O and radius equal sum of the radius of the given the radius of any one of the ci given radius.

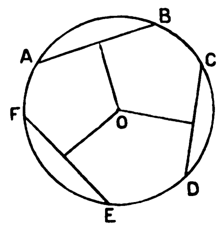


FIG. 275.

(7.) Let AB, CD, EF be any nun equal chords. They are therefore al distant from the centre O, that is, ⊥s on AB, CD, EF are all equal.

∴ the locus is a ⊙, with centre coi with the centre of the original ⊙, and ing all the chords.

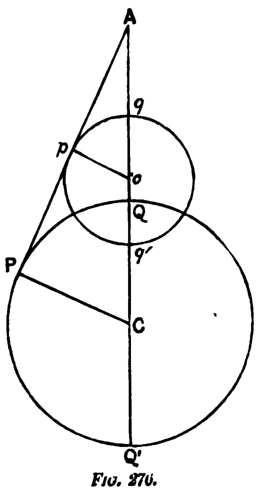


FIG. 276.

(8.) Let A be the given pt., anc centre of the given ⊙.

Draw $AQCQ'$, make $Aq = \frac{1}{2}$ A $Aq' = \frac{1}{2}$ AQ', and take c the mid of qq'.

Then

$Ac = \frac{1}{2}(Aq + Aq') = \frac{1}{4}(AQ + AQ') =$

Similarly, $cq = \frac{1}{2}$ CQ.

Draw AP to meet the ⊙ QPQ', a $Ap = \frac{1}{2}$ AP.

Then ∵ $Ap = \frac{1}{2}$ AP, and $Ac = \frac{1}{2}$.

∴ cp is ∥ to CP, and $cp = \frac{1}{2}$ C

Hence the locus of p is a ⊙, centre is c, and radius one-half radius of the given ⊙.

43. On the given base AB describe a segment of a ⊙, ADB, capable of containing an angle equal to the given angle E; this segment will be the locus of the vertex of the Δ.

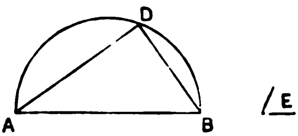


Fig. 277.

44. The locus is an equal circle, whose centre is moved parallel to the given line through a distance equal to the length of the line.

45. Let O be the middle pt. of the ladder AB, DC the vertical wall, EC the horizontal plane.

Then, ∵ ∠ ACB is a rt. ∠, a circle described with centre O and distance OA will pass through C.

∴ the locus is the quadrant of a ⊙ described with centre C and distance OA.

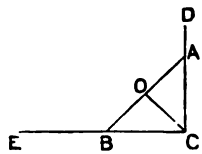


Fig. 278.

46. This is the well-known theorem that the locus is the chord of contact of tangents drawn from A.

Let FH be the chord of contact of tangents from A.

From O, the centre, draw $OP \perp$ to BC, and produce OP to meet HF produced in D.

D shall be the point of intersection of tangents from B and C.

For rect. OM, OA = sq. on OF, since OFA is a rt. ∠.

Also, since ∠s DPA, DMA are rt. ∠s, a ⊙ can be described about A, M, P, D.

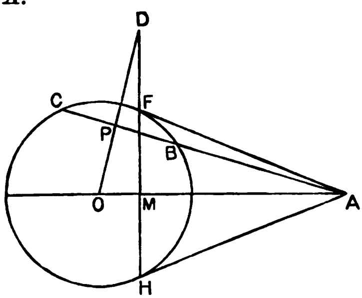


Fig. 279.

∴ rect. OP, OD = rect. OM, OA, (III. 36.)

= sq. on OF,

= sq. on OC;

∴ ∠ OCD is a rt. ∠, and CD touches the ⊙ at C.

Similarly BD touches the ⊙ at B.

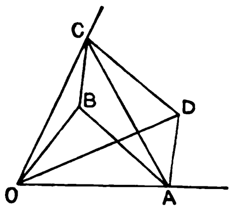


FIG. 280.

47. Since $\angle BCD = \angle COA$,

$\therefore$ sum of $\angle$s COA, CDA = two rt. $\angle$s;

$\therefore$ a circle may be described round $DCOA$; and this $\odot$ will be of constant magnitude, because AC, a line of constant magnitude, cuts off a segment containing a constant angle.

$\therefore$ the constant line CD cuts off a segment containing a constant angle,

$\therefore \angle COD$ is a constant angle.

and $\therefore$ D moves along a st. line passing through O.

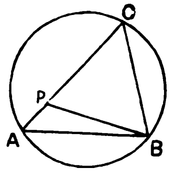


FIG. 281.

48. Since $\angle ACB$ is constant,

$\therefore \angle CPB$, which is equal to half the supplement of $\angle ACB$, is constant;

$\therefore \angle APB$ is constant;

$\therefore$ the locus is the segment of a $\odot$ described on AB, capable of containing an angle $= \angle APB$.

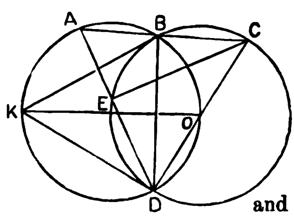


FIG. 282.

49. Let O be the centre of the $\odot$ $CBED$, and KO the diameter of $\odot$ ABD. Then KB, KD are $\perp$ to OB, OD, and $\therefore$ are tangents to the $\odot$ $CBED$.

Now

$\angle ABK = \angle ADK$ in same segment,

and $\angle ABK = \angle BDC$ in alternate segment,

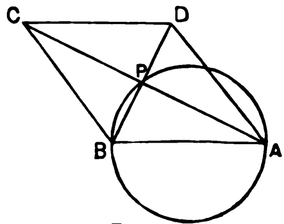
and $\angle ADK = \angle ECD$ in alternate segment,

$\therefore \angle BDC = \angle ECD$;

$= \angle EBD$ in the same segment;

$\therefore$ BE is $\parallel$ to CD.

Note.—In the diagram CD appears to pass through O, but this is *accidental*, and is not material to the proof.



50. Let $ABCD$ be the ▱, AB being the fixed diameter of the ⊙, and P the intersection of the diagonals.

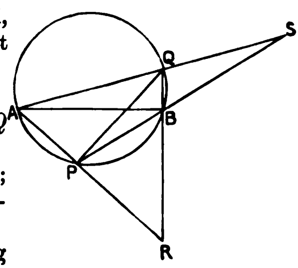
The angles at P are rt. angles (III. 31), and $AP=PC$, and BP is common to △s APB, CPB;

$\therefore AB=BC$.

$\therefore C$ lies on a circle whose radius is AB, and whose diameter is $2AB$

Fig. 283.

Similarly, D lies on the ○ce of an equal ⊙.



51. Let AB be the fixed diagonal, and PQ the diagonal of constant length.

Let AP, QB produced meet in R.

Then the angles APB and PBQ are constant.

$\therefore$ ∠s BPR and PBQ are constant;

$\therefore$ their difference, ∠ ARB, is constant;

$\therefore R$ lies on the ○ce of a ⊙ passing through A and B.

Fig. 284.

Similarly, S, the pt. of intersection of AQ, PB, lies on the ○ce of another circle.

BOOK IV.

Page 180.

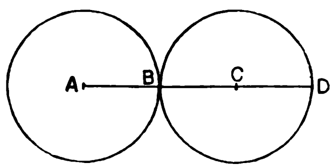


FIG. 285.

EXERCISE. Let A be the given pt. With centre A and radius AB = the given distance, describe a ⊙.

Produce AB to D, so that $BD=2AB$.

Bisect BD in C, and with centre C and radius CB describe a ⊙. This will be the circle reqd., and BD is ∴ the required diameter.

Page 181.

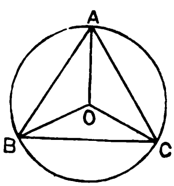


FIG. 286.

EXERCISE. Let ABC be an equilateral △, inscribed in a ⊙, of which the centre is O.

Then ∵ $OA=OB$, and $BC=AC$, and OC is common to the △s AOC, BOC,

$$\therefore \angle OCA = \angle OCB.$$

Similarly for the ∠s at A and B.

Page 183.

EXERCISE 1. For $AD=AF$, AO is common, and $OD=OF$;

$$\therefore \angle DAO = \angle FAO.$$

Ex. 2. Let DEF be a ⊙ inscribed in the right-angled △ ACB, which has ∠ ACB a rt. ∠, and let the ⊙ touch the sides of the △ in D, E, F.

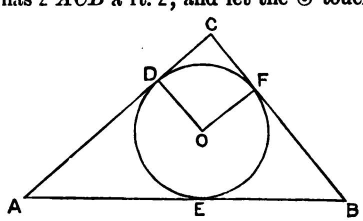


FIG. 287.

Draw the radii OD, OF; these are || to CF, CD, because the ∠s at D, C, F are all rt. angles.

Then

∵ $AD=AE$, and $BE=BF$,

$$\therefore AC+CB-AB=DC+CF,$$
$$=OD+OF,$$
$$=\text{diameter of } \odot.$$

Ex. 3. Let DEF be a ⊙ inscribed in the equilateral △ ABC, touching the sides in D, E, F.

Then ∵ ∠ OBE = ∠ OCE, (Ax. 7.)
and ∠ OEB = ∠ OEC,
and OE is common to △s OBE, OCE,
∴ $OB = OC$.
Similarly, $OA = OB = OC$.

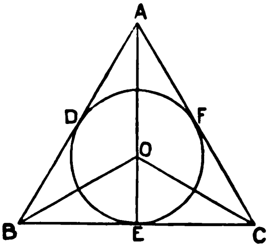


FIG. 288.

Ex. 4. Produce AB, AC, sides of the △ ABC to D, E.

Bisect ∠s DBC, ECB by BO, CO meeting in O.
Draw OM, ON, OP ⊥s to BD, CE, BC.
Then ∵ ∠ OBM = ∠ OBP,
and ∠ BMO = ∠ BPO, and OB is common,
∴ $PO = MO$, and similarly $PO = NO$.
∴ a ⊙ described with centre O and radius PO will touch BD, BC, CE.

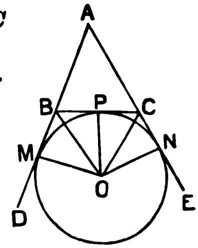


FIG. 289.

Page 185.

EXERCISE. About the ⊙ $ABCD$ describe the square $EFGH$, and draw its diagonals EG, HF, which intersect in O the centre. Then the four △s HOE, EOF, FOG, GOH are equal in all respects, and it is manifest from Eucl. IV. 4, that if a ⊙ be inscribed in each of the four △s, the ⊙s so described will be equal, and will touch one another and the ⊙ $ABCD$.

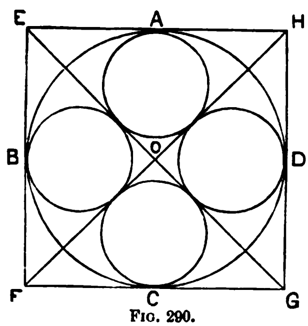


FIG. 290.

Page 186.

EXERCISE 1. From Ex. 2 on p. 144, it is clear that the ▱ must be a square or a rhombus, because in no other ▱ $ABCD$ is the sum of AB and CD *equal to the sum of* AD and BC.

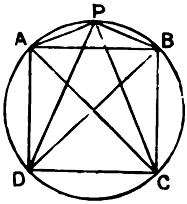


Fig. 291.

Ex. 2. Let $ABCD$ be a square inscribed in a ⊙. From P, any pt. in the Oce, draw PA, PB, PC, PD. Join AC, BD: these are diameters.

Then sum of sqq. on PA, PC = sq. on AC; (I. 47.)

and sum of sqq. on PB, PD = sq. on BD. (I. 47.)

∴ sum of sqq. on PA, PB, PC, PD = twice sq. on diameter.

Page 189.

Exercise. Since ∠ BAC = ∠ ACE, for they subtend equal arcs BC, AE,

∴ BA is parallel to CE.

Page 196.

Miscellaneous Exercises on Book IV.

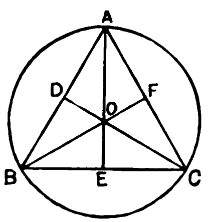


Fig. 292.

1. Let ABC be an equilateral △, and O the centre of the ⊙ described about it. CD, AE, BF the ⊥ s from the angular points on the opposite sides pass through O, and bisect the sides.

Then in the right-angled △s EOC, FOC,

∵ $FC = EC$ and OC is common,

∴ $OF = OE$; and similarly $OD = OF = OE$.

2. Let $ABCDEFGH$ be a regular octagon.

Bisect the ∠s HAB, ABC by AO, BO, meeting in O, and join OH.

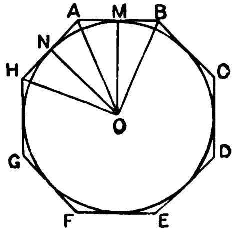


Fig. 293.

Then, as in IV. 13, we can show that ∠ AHO = ∠ ABO;

∴ OH bisects ∠ GHA,

and, if we draw OM, ON ⊥ s to AH, AB, we can show that $ON = OM$.

Similarly, if ⊥ s be drawn from O to the other sides of the octagon, these will all be equal.

∴ a ⊙, described with centre O and radius OM, will touch each side of the octagon.

3. Since BD touches the ⊙ ACD (see diagram to IV. 10),

$\therefore \angle BDC = \angle CAD$ in alternate segment.

Now $\angle BCD = \angle CBD = 2 \angle CAD$;

$\therefore \angle BCD = \angle CBD = 2 \angle BDC$;

$\therefore BDC$ is the triangle required.

4. Let $ABCD$ be the given rectangle.

The diagonals are equal, and bisect each other in O (I. 34, Ex. 1 and 2), and $\therefore$ a ⊙ described with centre O and radius OA will pass through A, B, C, D.

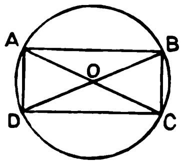


FIG. 294.

5. Describe a ⊙ about the isosceles △ ABC, which has $\angle BAC$ double of each of the angles at the base.

Then $\because \angle BAC =$ sum of $\angle$s ABC, ACB,

$\therefore \angle BAC$ is a right angle;

and $\therefore BC$ is the diameter of the ⊙ described about △ ABC.

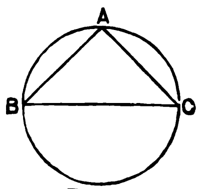


FIG. 295.

6. Let ABC be an equilateral △ described about the ⊙ DEF, touching the ⊙ at the points D, E, F.

Join DE, EF, FD.

Then $\because AD = AE$,

$\therefore \angle ADE = \angle AED$.

Now $\angle DAE = \frac{1}{3}$ of two rt. $\angle$s,

$\therefore \angle ADE$ is $\frac{1}{3}$ of two rt. $\angle$s,

and ADE is an equilateral △.

$\therefore DE$ is $\parallel$ to BC.

Also, AC is bisected in E,

and $\therefore BC = 2 . DE$.

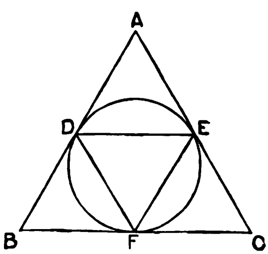


FIG. 296.

7. Let $ABCD$ be a quadrilateral figure, whose diagonals intersect in P, and let rect. AP, PC = rect. BP, PD.

From M, N the middle pts. of AC, BD, draw OM, $ON \perp$ to AC, BD, meeting in O.

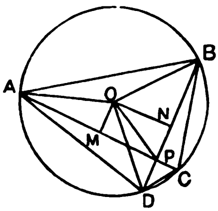


FIG. 297.

Join OA, OB, OP, OD.

Then

rect. AP, PC with sq. on MP = sq. on AM;

and $\therefore$ rect. AP, PC with sqq on MP, OM = sqq. on AM, OM;

$\therefore$ rect. AP, PC with sq. on OP = sq. on OA.

So also,

rect. BP, PD with sq. on OP = sq. on OB.

$\therefore OA = OB$.

Now $OD = OB$ (I. 4), and similarly $OA = OC$,

$\therefore$ a ⊙ described with centre O and radius OA will pass through A, B, C, D.

8. Let ABC be an equilateral △ inscribed in a ⊙, of which O is the centre.

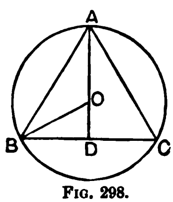


FIG. 298.

Produce AO to D, then AD is $\perp$ to BC.

Then $\angle OBD = \frac{1}{2} \angle ABC = \frac{1}{3}$ of a rt. $\angle$;

$\therefore \angle BOD$ = twice $\angle OBD$;

$\therefore BO$ = twice OD. (Ex. 3, p. 116.)

Now

sq. on AB = sq. on AO + sq. on BO + 2 rect. AO, OD, (II. 12.)

= sq. on AO + sq. on AO + rect. AO, BO,

= three times sq. on AO.

And AO is equal to the side of the hexagon inscribed in the ⊙.

9. Let $ABCD$ be the given rhombus.

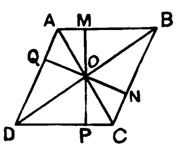


FIG. 299.

Draw the diagonals AC, BD, and from O the point of intersection draw OM, ON, OP, $OQ \perp$ to the sides.

Then $\because \angle OMB = \angle ONB$, and $\angle OBM = \angle OBN$,

and OB is common to the △s OMB, ONB;

$\therefore OM = ON$.

Similarly, $OM = OQ = OP$.

$\therefore$ a ⊙ described with centre O and radius OM will touch the sides.

10. Let O be the centre of the ⊙.

Join AO, BO, and let MO cut AB in P.

Then rt. $\angle MAO =$ rt. $\angle MBO$, and $MA = MB$, and $AO = BO$,

$$\therefore \angle AMO = \angle BMO.$$

Hence $BP = AP$, and the $\angle$s at P are rt. $\angle$s;

$\therefore$ COP is a straight line, because the line drawn from an angular pt. of the equilateral $\triangle$ through the centre of the ⊙ described about the $\triangle$ bisects the opposite side at rt. angles.

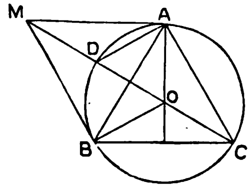


FIG. 300.

Next,

since $\angle MAB = \angle ABC = \frac{1}{3}$ of two rt. $\angle$s,

$\therefore \angle MBA = \frac{1}{3}$ of two rt. $\angle$s,

and $\therefore \angle AMB = \frac{1}{3}$ of two rt. $\angle$s;

$\therefore \angle AMD = \angle ACD$; (Ax. 7.)

and $\angle MAD = \angle ACD$ in alternate segment,

$\therefore \angle AMD = \angle MAD$, and $\therefore AD = MD$.

And $\because \angle AMD = \frac{1}{3}$ of a rt. $\angle$,

$\therefore \angle MAD = \frac{1}{3}$ of a rt. $\angle$;

and $\therefore \angle DAO = \frac{2}{3}$ of a rt. $\angle$,

$= \angle AOD$;

$\therefore DA = DO$;

$\therefore MD = DO = OC$.

11. Let A, B, C, D, E, F be the angular pts. of the regular hexagon inscribed in the ⊙. Then these are the pts. in which the sides of the circumscribed hexagon touch the ⊙. Take O the centre. Then since the $\angle$s OAN, OBN are right $\angle$s, a circle can be described about $ANBO$, of which ON is the diameter.

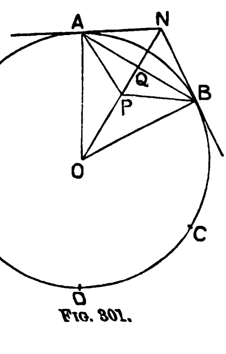


FIG. 301.

Bisect ON in P.

Now AOB is an equilateral $\triangle$, and ON bisects $\angle AOB$ (I. 8), and is $\perp$ to AB.

Hence $PA=PB$ (I. 4), and $\angle APO = \angle BPO$,
and $\therefore \angle APQ = \angle BPQ$, and each of these $\angle s = 2 \angle AOP = \frac{2}{3}$ of a r[t. ∠]

$$\therefore \angle PAQ = \tfrac{1}{3} \text{ of a rt. } \angle = \angle PAO\,;$$

$$\therefore \angle QAN = \tfrac{1}{3} \text{ of a rt. } \angle.$$

Hence, since PAN is an equilateral $\triangle$, AQ bisects PN.
Then area of external hexagon $= 12 . \triangle AON = 6 .$ rect. AQ, O[N]
and area of internal hexagon $= 12 . \triangle AOQ = 6 .$ rect. AQ, O[Q]
and $\therefore$, since $OQ = \frac{3}{4} ON$,
area of internal hexagon $= \frac{3}{4}$. area of external hexagon.

12. Take AB as the diameter of the semicircle, and bisect it in [C].
Take $DE=5\ BC$, and on DE describe a rectangle $DEMR$ = sq. [on] BC.

Take in DE the part $DF=BC$, and draw $FG \parallel$ to DR. Th[en] $DFGR = \frac{1}{5}$ sq. on BC.

Describe a square S = rectangle $DFGR$, and in CB take CH = a si[de] of this square.

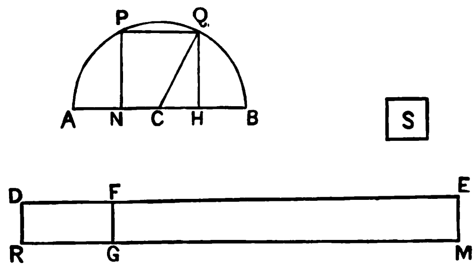


FIG. 302.

Draw $HQ \perp$ to CB.

$$\text{Then sq. on } CQ = 5 \text{ sq. on } CH,$$

$$\therefore \text{sq. on } HQ = 4 \text{ sq. on } CH, \qquad \text{(I. 47.)}$$

$$\therefore HQ = 2\ CH.$$

Take $CN=CH$, and draw $NP \perp$ to AC.

$$\text{Then } PN=QH, \text{ and } \therefore QP=NH.$$

Then $PNHQ$ is the square required.

13. Let A, B, C, D, E, F be the angular pts. of a regular hexagon inscribed in the given ⊙, whose centre is O.

Produce OD to M, so that $DM=DO$; and produce OE to N, so that $EN=EO$.

Join MN, bisect it in P, and join OP.

Then since $\angle EOD=\frac{1}{3}$ of two rt. ∠s, and $ON=OM$, $\therefore$ ONM is an equilateral △, and $\therefore$ OP is $\perp$ to MN.

$\therefore$ circles described with centres M, N, and radius $=DO$, will touch each other at P, and will touch the given ⊙ at D and E.

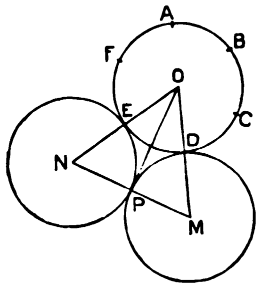


Fig. 303.

Similarly, by producing OF, OA, OB, OC the four other ⊙s may be described.

14. Let OM, ON, OP be the three perpendiculars, so placed that $\angle MON$ is the supplement of one of the given angles, and $\angle PON$ the supplement of another of the angles; then $\angle POM$ must be the supplement of the third angle.

Draw AC, CB, BA $\perp$ to OM, OP, ON, meeting in A, B, C. Then ABC is the triangle required, because the ∠s at A, B, C are supplementary to ∠s MON, NOP, POM respectively.

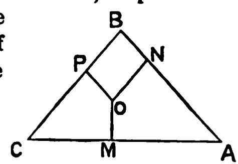


Fig. 304.

15. Let AB, AC be the given lines.

Bisect $\angle BAC$ by AE. Draw $AD \perp$ to AB, and make $AD =$ the given radius.

Draw $DO \parallel$ to AB, and let it meet AE in O, and draw $OQ, OR \perp$ to AB, AC.

Then $\because$ $\angle OQA=\angle ORA$, and $\angle QAO=\angle RAO$, and AO is common;

$$\therefore OR=OQ=AD.$$

$\therefore$ a ⊙ described with centre O and radius $=AD$, will touch AB, AC in Q, R.

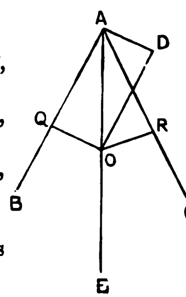


Fig. 305.

16. Let O, C be the centres of the two circles.

Draw OP a tangent to the other ⊙, then since the ⊙s cut one another at rt. ∠s, P is a pt. in the Oce of the circle whose centre is O.

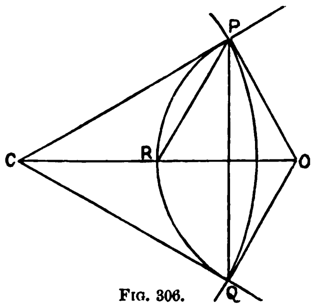


FIG. 306.

Draw OQ, the other tangent, from O to the larger ⊙.

Then since $OC=2.OP$, and OPC is a right ∠, ∴ OC is bisected by the smaller ⊙ in R.

∴ R is the centre of a ⊙ passing through $CPOQ$.

∴ $RP=RO=OP$, and ∴ PRO is an equilateral △.

∴ ∠ $POR=\frac{1}{3}$ of two rt. ∠s ;

∴ ∠ $POQ=\frac{1}{3}$ of four rt. ∠s ;

and ∠ $PCQ=\frac{1}{3}$ of two rt. ∠s ;

∴ PQ is the side of a regular hexagon in the larger circle, and PQ is the side of an equilateral △ in the smaller circle.

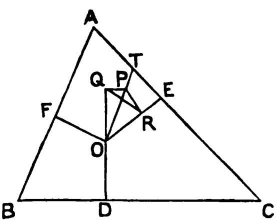


FIG. 307.

17. Let OP meet AC in T.

The angles at Q and R being right ∠s, a ⊙ will circumscribe $PQOR$.

∴ ∠ OQR = ∠ OPR = ∠ OTE = ∠ BAC.

Also, ∠ QOR = supplement of ∠ EOD,
= ∠ ACB ;

∴ the third ∠ ORQ = ∠ ABC ;

∴ △s OQR, ABC are equiangular.

Page 198.

Senate-House Riders.

1848. I. 34. Let $ABCD$ be a ▱, and let the diagonals intersect each other in E.

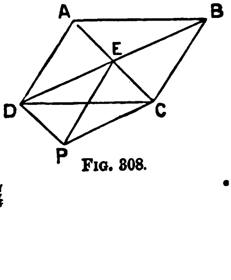


FIG. 308.

Then △ ADC = half the ▱,
= △ DAB.

Take from each △ AED.

Then △ DEC = △ BEA.

Similarly △ AED = △ BEC.

Next, draw DP ∥ to EC and CP ∥ to ED, and join EP.

Then $DP = EC = EA$, and ED is common to $\triangle$s PDE, ADE, and $\angle EDP = \angle AED$;

$$\therefore \triangle DEP = \triangle AED.$$

But $\triangle DEP$ = half the $\square$ $DECP = \triangle DEC$.

$$\therefore \triangle DEC = \triangle AED = \triangle BEC = \triangle BEA.$$

The diagonal bisects the angles when all the sides are equal.

III. 15. Let AB, CD be any two equal chords in a circle. Then they are equidistant from O the centre. Draw $\perp$s OM, ON to AB, CD. Then a $\odot$ described with centre O and distance OM will touch each of the equal chords.

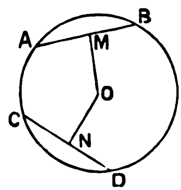


FIG. 309.

III. 20. $\angle AOC$ = twice $\angle ADC$;
$\angle BOD$ = twice $\angle BAD$;
$\therefore$ sum of
$\angle$s AOC, BOD = twice sum of $\angle$s ADC, BAD
= twice $\angle AEC$.

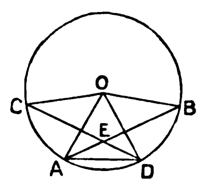


FIG. 310.

Page 199.

1849. I. 1. Let AB be the given line.

With centre A and distance AB describe a $\odot$, and produce BA to meet this $\odot$ in D.

With centre B and distance BA describe a $\odot$, and produce AB to meet this $\odot$ in C.

Then with distance A and radius AC describe a $\odot$, and with distance B and radius BD describe a $\odot$, and from E, where these $\odot$s intersect, draw the st. lines EA, EB.

Then EAB is the $\triangle$ required.

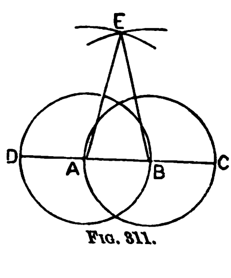


FIG. 311.

II. 11. Taking the diagram on p. 89, it is clear that

(1.) rect. *DC*, *CK* = sq. on *DK*.

(2.) rect. *DF*, *FA* = sq. on *AD* (since *AD* = *AB*).

(3.) rect. *KG*, *GH* = sq. on *HK*.

(4.) *BE* is similarly divided, but the proof depends o Book VI., since if *M* be the pt. where *BE* cu *HK*, △s *BHM*, *BAE* are similar, and ∴ *AB* an *BE* are similarly divided in *H* and *M*.

IV. 4. For this problem of the escribed circle see IV. 4, Ex. 4.

1850. I. 34. See Exercises 6, 5, 2 on p. 59.

II. 14. See Exercise 50 on p. 120.

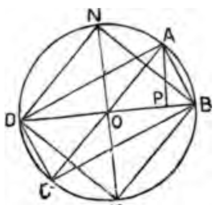


FIG. 312.

III. 31. Let *ABCD* be any rectangle in scribed in a ⊙; then the diagonals bisec each other in the centre *O*.

Draw the diameter *NOM* ⊥ to *BD*; the *NBMD* is a square, and its area is = rect. *N*(*BD*. Draw *AP* ⊥ to *BD*.

Now area of rectangle *ABCD* = *AP*, *BD* and since *AP* is less than *AO*, *AP* is less tha *NO*, and ∴ area of the square is greater tha area of the rectangle.

III. 34. Construct an equilateral △ *ABC*, and draw *AD* ⊥ to *BC*.
Then ∠ *BAD* = $\frac{1}{6}$ of two rt. ∠s.
Draw *HEG* a tangent to the ⊙ *MEF*.
Make ∠ *FEG* = ∠ *BAD*.

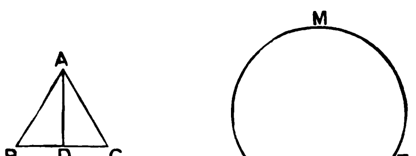


FIG. 313.

Then ∠ *FEG* = $\frac{1}{6}$ of two rt. ∠s,
and ∠ *FEH* = $\frac{5}{6}$ of two rt. ∠s;
∴ ∠ in segment *FNE* = five times ∠ in segment *FME*.

IV. 10. If O be the centre of the smaller ⊙,

$\angle COD = 2 \angle CAD$.

Now $\angle CAD = \frac{1}{5}$ of two rt. $\angle$s;

$\therefore \angle COD = \frac{1}{5}$ of four rt. $\angle$s.

Hence CD, to which BD is equal, is the side of a regular pentagon inscribed in the smaller ⊙.

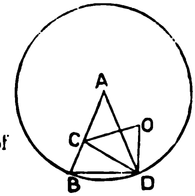


FIG. 314.

1851. I. 38. Let ABC, ABD be two equal △s upon the same base AB, and on opposite sides of it: join CD meeting AB in E.

Make $\angle ABN = \angle ABD$, and $BN = BD$, and join CN, AN, EN.

Then $\triangle ABN = \triangle ABD = \triangle ABC$.

Hence, by applying the Proposition as in I. 39 and 40, we obtain that CN is ∥ to BA.

$\therefore \triangle CEB = \triangle NEB$, on same base,

$= \triangle DEB$, by the construction of the figure.

Hence △s CEB, DEB must be on equal bases, CE, DE.

For, if not, let $CE = EF$.

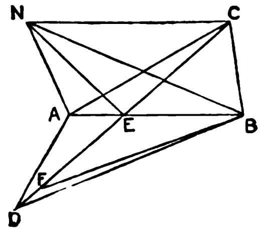


FIG. 315.

Then $\triangle CEB = \triangle BEF$ (I. 38), which is impossible;

$\therefore CE = DE$.

I. 47. See Exercise 28 on p. 118.

III. 22. From any angular pt. A_1 draw $A_1 A_4$, $A_1 A_8$ to the *third* angular pt. on each side of A_1, cutting off quadrilaterals. Since the number of remaining sides between A_4, A_8 is even, a certain number of quadrilaterals may be formed by joining every second angular pt. from A_4 with A_1, and all these quadrilaterals are inscribed in the ⊙.

$\therefore \angle A_1A_2A_3 + \angle A_3A_4A_1 =$ two rt. angles,

$\angle A_1A_4A_5 + \angle A_5A_6A_1 =$ two rt. angles,

$\angle A_1A_6A_7 + \angle A_7A_8A_1 =$ two rt. angles,

$\angle A_1A_8A_9 + \angle A_9A_{10}A_1 =$ two rt. angles.

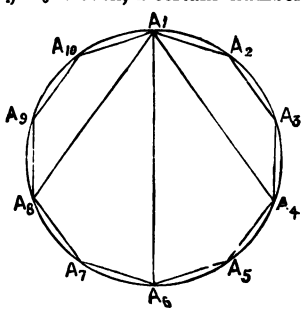


FIG. 316.

Adding, sum of alternate angles A_2, A_4, A_6, etc. = twice as many rt. angles as there are quadrilaterals = $2\ (r-1)$ rt. angles, the number of sides of the polygon being $2r$.

$\therefore$ sum of alt. angles + two rt. $\angle$s = as many rt. $\angle$s as there are sides.

IV. 16. Describe a regular quindecagon in a $\odot$.

Each of its angles = $\frac{13}{15}$ of two rt. $\angle$s. (I. 32, Cor. 1.)

Produce one of the sides; then the angle between the produced part and the adjacent sides = $\frac{2}{15}$ of two rt. $\angle$s.

Take a second angle four times as great as this angle, and therefore = $\frac{8}{15}$ of two rt. $\angle$s.

Take a third angle = one of the angles of an equilateral $\triangle$, and therefore = $\frac{5}{15}$ of two rt. $\angle$s.

Then describe a $\triangle$ with its angles equal to these angles, and then inscribe in the given $\odot$ a triangle with its angles equal to the angles of this $\triangle$.

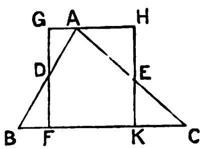


Fig. 317.

1852. I. 42. Let ABC be the given $\triangle$.

Bisect AB in D, and AC in E.

Draw GDF making $\angle\ GFC$ equal to one of the given angles. Through A draw $GAH \parallel$ to BC, and through E draw $HEK \parallel$ to GF.

Then since $\triangle\ ADG = \triangle\ BDF$ in all respects;

and $\triangle\ EHA = \triangle\ EKC$ in all respects;

the lines DF, EK divide the triangle in the required manner.

II. 12. Let BC be the base, ABC one of the $\triangle$s.

Draw $AD \perp$ to BC, or BC produced.

Then since (fig. 1) sq. on AB = sqq. on AC, BC with twice rect. BC, CD;

Fig. 1.

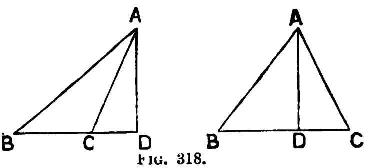


Fig. 2.

Fig. 318.

$\therefore$ since difference of sqq. on AB, AC is constant, and BC is constant,

$\therefore$ CD is constant;

$\therefore$ the vertex lies in the line drawn from $A \perp$ to BC produced.

Similarly (fig. 2), it may be shown that, in the case of $\angle$ at C being acute $\angle$, that the vertex lies in the $\perp$ on BC from A.

We have taken AB greater than AC, and if we take AC greater than AB, we can show in the same way that the vertex lies in another fixed st. line.

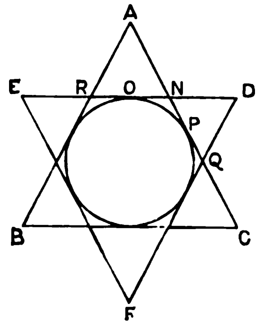


FIG. 319.

IV. 3. Let ABC, DEF be the $\triangle$s; O, P two of the pts. of contact; R, N, Q three pts. of intersection of the sides.

Then $DO=\frac{1}{2}DE=\frac{1}{2}AC=AP$,

and $NO=NP$,

and $\therefore AN=DN$.

Also, $\angle ANR = \angle DNQ$,

and $\angle NAR = \angle NDQ$,

$\therefore NR=NQ$. (I. 26.)

Similarly, the other sides of the hexagon may be proved to be equal.

When ED is $\parallel$ to BC, $\angle ARN = \angle ANR$;

and $\therefore \angle NRB = \angle RNC$;

and the hexagon is equiangular; but not otherwise.

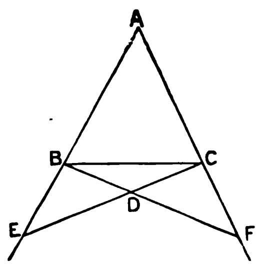


FIG. 320.

1853. I. B. COR. Let ABC be an isosceles $\triangle$, having $AB = AC$. Draw BDF, CDE as directed.

Then $\triangle BDC$ is evidently isosceles.

Also,

$\angle DBE + \angle BDE + \angle BED=$two rt. $\angle$s;

and $\angle DBE + 4\angle CBD=$two rt. $\angle$s;

and $\angle BDE=2\angle CBD$;

$\therefore \angle BED=2\angle CBD = \angle BDE$;

$\therefore \triangle BDE$ is isosceles; and so is $\triangle CDF$.

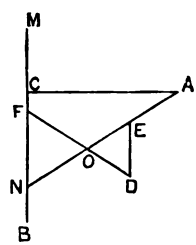


FIG. 321.

I. 29. Let A, D be the pts. and BM the line.

Draw $AC \perp$ to CB.

Make $\angle CAN=\frac{1}{3}$ of a rt. $\angle$.

Then $\angle CNA=\frac{2}{3}$ of a rt. $\angle$.

Draw $DE \parallel$ to BC.

Make $\angle EDF=\frac{2}{3}$ of a rt. $\angle$;

$\therefore \angle OFN=\frac{2}{3}$ of a rt. $\angle$;

$\therefore FON$ is an equilateral $\triangle$.

II. 11. Since $FA = AH$, and $AB = AD$, and $\angle FAB = \angle DAH$,
$\therefore \angle AFB = \angle AHD$;
$\therefore \angle LHB = \angle AFB$, and $\angle LBH = \angle FBA$,
$\therefore \angle BLH = \angle FAB =$ a rt. $\angle$.

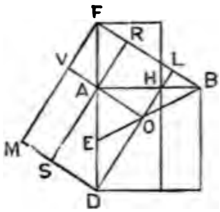


FIG. 322.

Next, $EB = EF$, and $\therefore \angle EBF = \angle EFB$.
$\therefore$ in right-angled $\triangle$s FLD, OLB,
$\because \angle FLD = \angle OLB$, and $\angle DFL = \angle OBL$,
$\therefore \angle FDL = \angle LOB$;
$\therefore \angle EDO = \angle EOD$, and $\therefore EO = ED = EA$,
and $\therefore \angle AOD$ is a rt. $\angle$.
Again, since $EB = EF$, and $EO = EA$,
$\therefore OB = AF = AH$.
$\therefore$ in $\triangle$s AOH, OLB, since $\angle OAH = \angle LOB$,
and $\angle AOH = \angle OLB$, and $AH = OB$;
$\therefore AO = LO$, and $HO = LB$.

Also, $DH = FB$, and $HO = LB$, and $\therefore DO = FL$. Complete the rectangles $DLFM$, $DLRAS$.

Then rect. DL, $LO =$ fig. $RD =$ fig. $VD =$ sq. on DO.

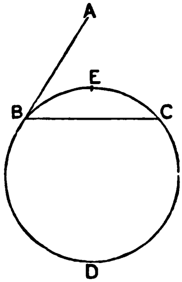


FIG. 323.

III. 32. Let A be the given pt., BCD the circle, 2θ the given angle. Draw AB a tangent to the $\odot$ at B, and make $\angle ABC =$ a rt. angle $- \theta$.

Then angle in segment $BDC =$ a rt. angle $- \theta$;
and angle in segment $BEC =$ a rt. angle $+ \theta$;
$\therefore$ difference of the angles in the segments $= 2\theta$.

Then if in $\odot$ BCD a concentric $\odot$ be described touching BC, and a tangent to the inner $\odot$ be drawn from A, the part of this tangent intercepted by $\odot$ BCD will cut off the required segments.

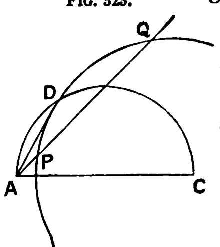


FIG. 324.

III. 36. Let C be the given centre, and A the fixed point in the given line APQ.

Describe a semicircle on AC as diameter, and place in it $AD =$ side of given square.

With centre C and radius CD describe a circle cutting the given line in P, Q. This $\odot$ touches AD, because $\angle ADC$ is a right angle.
$\therefore$ rect. AP, $AQ =$ sq. on AD.

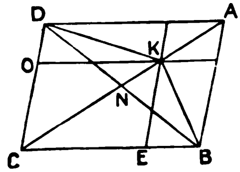


FIG. 325.

1854. I. 43. Join DB.

Then $KC - KA$
$= OB - OA$,
$= 2(\triangle CKB - \triangle AKD)$,
$= 2(\triangle CNB + \triangle NKB - \triangle AND + \triangle DKN)$,
$= 2 \triangle BKD$.

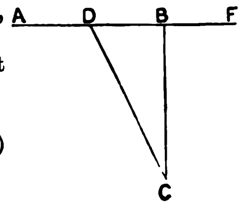


FIG. 326.

II. 11. Let AB be the given line.

Bisect AB in D, and draw $BC = AB$, and $\perp$ to AB.

Join DC, and produce AB to F, so that $DF = DC$.

Then rect. AF, FB with sq. on DB
$= $ sq. on DF, (II. 6.)
$= $ sq. on DC,
$= $ sqq. on CB, BD;
$\therefore$ rect. AF, $FB = $ sq. on CB,
$= $ sq. on AB.

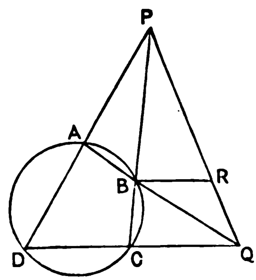


FIG. 327.

III. 22. Let the $\odot$s described about ABP, BCQ meet in R.

Then $\angle PRB = $ supplement of $\angle PAB$,
$= \angle DAB$.

And $\angle QRB = $ supplement of $\angle QCB$,
$= \angle DCB$.

Now sum of $\angle$s DAB, $DCB = 2$ rt. $\angle$s;
$\therefore$ sum of $\angle$s PRB, $QRB = 2$ rt. $\angle$s;
$\therefore$ PRQ is a straight line.

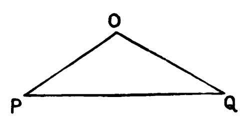


FIG. 328.

IV. 10. Let PQ be the given st. line.

Make $\angle PQO = \angle BAD$ in the Proposition,

and $\angle QPO = \angle BAD$.

Each of these $\angle$s is $\frac{1}{5}$ of two rt. $\angle$s;
$\therefore$ $\angle POQ = \frac{3}{5}$ of two rt. $\angle$s;
$\therefore$ *$\angle POQ$ is treble of* each of the angles at the base.

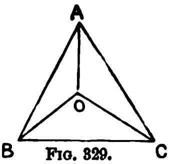


FIG. 329.

1855. I. 20. Let O be the point.
Then sum of OA, OC is greater than AC;
sum of OC, OB is greater than BC;
sum of OB, OA is greater than BA;
$\therefore$ sum of OA, OB, OC is greater than half the sum of AB, BC, CA.

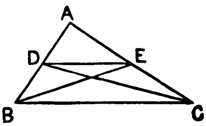


FIG. 330.

I. 47. Let DE be drawn $\parallel$ to BC the hypotenuse of the right-angled $\triangle ABC$. Join BE, CD.
Then sq. on DC = sum of sqq. on DA, AC,
and sq. on BE = sum of sqq. on BA, AE;
$\therefore$ sum of sqq. on DC, BC = sum of sqq. on BA, AC, DA, AE,
= sum of sqq. on BC, DE.

II. 9. See Exercise 41 on p. 119.

III. 27. Let BAC be any one of the $\triangle$s.
Draw BDP, CDO $\perp$s on AC, AB.
Then since $\angle$s at O, P are rt. $\angle$s, $\angle BDC = \angle ODP$,
= supplement of $\angle BAC$,
$\therefore \angle BDC$ is a constant $\angle$.

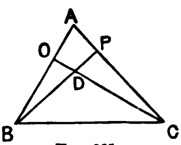


FIG. 331.

$\therefore$ locus of D is the segment of a $\odot$ described on BC, capable of containing $\angle BDC$.
Also, the line bisecting $\angle BDC$ must pass through the centre of the $\odot$ described about $\triangle BDC$, and similarly for the lines bisecting $\angle BDC$ in other positions.

IV. 4. Join DG, EG, and draw GK, GH $\perp$s to AD, AE.

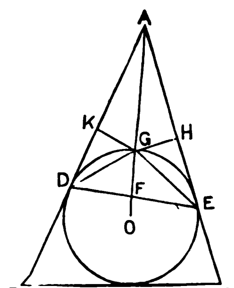
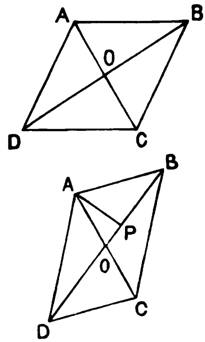


FIG. 332.

Let O be the centre of the $\odot$ inscribed in $\triangle ABC$.
Then $\because AE = AD$, and AF is common, and $\angle DAF = \angle EAF$,
$\therefore GF$ bisects DE at rt. $\angle$s;
$\therefore GD = GE$, and $\angle GEF = \angle GDF$,
$\therefore \angle GDK = \angle GEH$, and $\therefore GK = GH$.
Also, $\angle HEG = \angle GDE$, (III. 32.)
$= \angle GEF$;
$\therefore GF = GH$.
Hence a $\odot$ described with centre G and distance GF will touch the sides of $\triangle ADE$.



1856. I. 34. Let $ABCD$ be a rhombus, whose diameters AC, BD intersect in O. Then ∠s at O are rt. ∠s.

Then area of rhombus = rect. AO, BD.

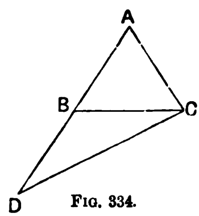
Now let $ABCD$ be any ▱ not a rhombus, having its diagonals equal to the diagonals of the rhombus.

Then, if AP be drawn ⊥ to BD,

area of ▱ $ABCD$ = rect. AP, BD;

and AP is less than AO, ∴ area of the ▱ is less than area of the rhombus.

FIG. 333.



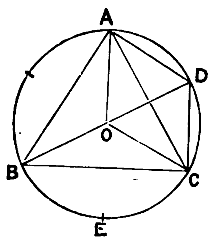
II. 12. Sum of sqq. on AC, CD = twice sum of sqq. on CB, BA (p. 91, Ex.);

∴ sum of sqq. on AB, CD = twice sum of sqq. on CB, BA;

∴ sq. on CD = twice sq. on CB with sq. on BA.

FIG. 334.

IV. 15. Let ABC be an equilateral △ inscribed in the ⊙ of which O is the centre.



Produce BO to meet the Oce in D, and join AD, DC.

Then ∠ AOD = twice ∠ OAB,

$= \frac{1}{3}$ of two rt. ∠s.

∴ AD is the side of a regular hexagon inscribed in the ⊙;

and since △ AOD is clearly equilateral, AD = radius of circle.

Similarly, DC may be shown to be a side of this hexagon.

FIG. 335.

And if AO, CO produced meet the Oce in E, F; CE, EB, BF, FA are sides of the hexagon.

Also area of hexagon = six times △ AOD,

= six times △ AOB,

= twice △ ABC.

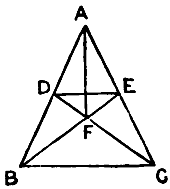


FIG. 336.

1857. I. 35. Sum of $\triangle$s EFC, $EFD = \triangle EDC$ $= \frac{1}{2} \triangle ADC = \frac{1}{4} \triangle ABC$;

sum of $\triangle$s DFB, $EFD = \triangle DEB$ $= \frac{1}{2} \triangle ABE = \frac{1}{4} \triangle ABC$;

$\therefore \triangle EFC + \triangle DFB + 2\triangle EFD = \frac{1}{2} \triangle ABC$;

$\therefore \triangle AFE + \triangle AFD + 2\triangle EFD = \frac{1}{2} \triangle ABC$;

$\therefore \triangle ADE + 3\triangle EFD = \frac{1}{2} \triangle ABC$.

Also, $\triangle ADE = \frac{1}{4} \triangle ABC$;

$\therefore 3\triangle EFD = \frac{1}{4} \triangle ABC$;

$\therefore \triangle ADE = 3\triangle EFD$.

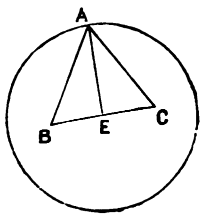


FIG. 337

II. 13. Let ABC be the $\triangle$, E the centre of the $\odot$, which is of constant radius.

Then since sum of sqq. on BA, AC=twice sum of sqq. on AE, BE; (p. 91, Ex.)

$\therefore$ the sum of sqq. on BA, AC is a constant.

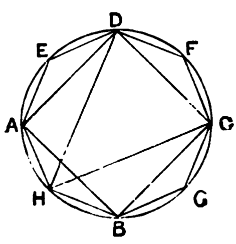


FIG. 338.

III. 22. Let $ABCD$ be the quadrilateral, and take E, F, G, H pts. in the exterior segments. Complete the figure as in the diagram.

Then sum of $\angle$s AED, AHD=two rt. $\angle$s (III. 22.)

and sum of $\angle$s DHC, DFC=two rt. $\angle$s (III. 22.)

and sum of $\angle$s CHB, CGB=two rt. $\angle$s (III. 22.)

$\therefore$ sum of $\angle$s AED, AHB, DFC, CGB=six rt. $\angle$s.

IV. 4. Let ABC be the $\triangle$; AD, BE, CF the $\perp$s;
d_1, d_2 the diameters of $\odot$s inscribed in $\triangle$s ABD, ADC;
d_3, d_4 the diameters of $\odot$s inscribed in $\triangle$s BEC, BEA;
. the diameters of $\odot$s inscribed in $\triangle$s ACF, BCF.

Then by Ex. 2 on p. 183,

$$AB+d_1=AD+BD,$$
$$AC+d_2=AD+DC,$$
$$BC+d_3=BE+CE,$$
$$AB+d_4=BE+EA,$$
$$AC+d_5=CF+FA,$$
$$BC+d_6=CF+FB;$$

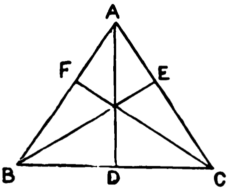


FIG. 339.

∴ adding, we obtain

sum of diameters + twice sum of sides = twice sum of ⊥ s + sum of sides;

∴ sum of diameters + sum of sides = twice sum of ⊥ s.

1858. I. 28. The rider has been already explained on p. 47.

II. 7. Let AB be the given line.

Draw BD ⊥ to AB, and equal to AB.

Join AD. Produce AB to C, so that $AC=AD$.

Then sqq. on AC, $BC=2$ rect. AC, CB with sq. on AB;

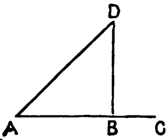


FIG. 340.

∴ sqq. on AB, BD, $BC=2$ rect. AC, CB with sq. on BD;

∴ sqq. on AB, $BC=2$ rect. AC, CB.

III. 19. Let A be the centre of the given ⊙, B the given pt. in the st. line CD.

Draw BE ⊥ to CD. Join BA, and produce it to P so that rect. BA, $AP=$ sq. on radius of given ⊙.

Then P is known, and it is a point in the Oce of the ⊙ which has to be described. Bisect BP in O, and draw OQ ⊥ to BP, and meeting BE in Q, then Q will be the centre of the ⊙ passing through P and touching CD in B.

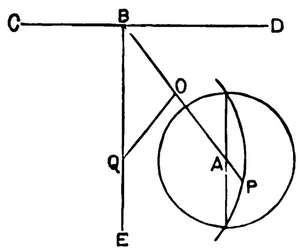


FIG. 341.

Also, since rect. BA, $AP=$ sq. on radius of original ⊙, the common chord of the two ⊙s will evidently pass through A, and the new ⊙ will therefore bisect the original ⊙.

1859. I. 41. Let $ABCD$ be the given ▱.

Take $EC=$ one-third of BC,

$FC=$ one-third of DC,

and join AE, AF, and draw EG ∥ to AB.

Then △ $ABE=\frac{1}{2}$ ▱ AE,

$=\frac{1}{3}$ ▱ $ABCD$;

and similarly △ $ADF=\frac{1}{3}$ ▱ $ABCD$;

∴ *quadrilateral* $AECF=\frac{1}{3}$ ▱ $ABCD$.

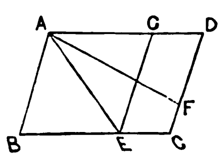


FIG. 342.

II. 13. See Exercise 10 on p. 94.

III. 31. Let ABC, ADE be the equal ⊙s.
Draw the chords AB, AD, at rt. ∠s to each other.
Draw $EOAFC$ passing through the centres O, F.

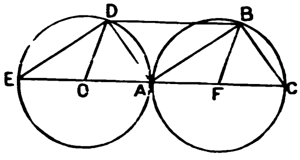


FIG. 343.

Then ∠s ABC, ADE are rt. ∠s, (III. 31.)

and $\angle DAO + \angle BAC =$ a rt. ∠,

$= \angle BCA + \angle BAC$,

$\therefore \angle DAO = \angle BCA$, and $\therefore DA$ is ∥ to BC.

Again, in △s EDA, ABC,

$\because \angle EDA = \angle ABC$, and $\angle DAE = \angle BCA$, and $EA = AC$,

$\therefore AD = BC$;

and $\therefore DB$ is ∥ and = to AC (I. 33); $\therefore DB = OF$.

IV. 4. Let BC be the base; BAC the vertical ∠; O the centre of one of the escribed ⊙s, touching AC, and the other sides produced.

Join OB, OC, and produce BC to T.

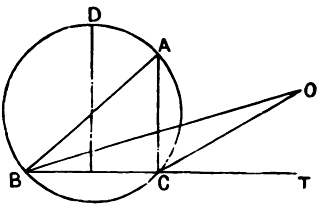


FIG. 344.

Then OB, OC bisect ∠s ABC, ACT.

Now $\angle BOC = \angle OCT - \angle OBC$,

$= \frac{1}{2} \angle ACT - \frac{1}{2} \angle ABC$

$= \frac{1}{2} \angle BAC$.

The locus is therefore the segment of a ⊙ passing through B and C, and having $\frac{1}{2} \angle BAC$ as the angle in it. The angle subtended by BC at the centre of this ⊙ being $= \angle BAC$, the centre must lie on the circumference of the circle described about BAC, and as BC is a chord, the centre must be at D, the point farthest from BC.

Note.—If A move to the other side of BC, the conditions are *different*, and the locus will be the centre of another ⊙.

1860. I. 35. Draw $CP \parallel$ to BD.
Then $\because \angle MBD = \angle MCP$,
and $\angle DMB = \angle PMC$,
and $BM = MC$, $\therefore CP = BD$.
Now $\angle AED = \angle ADE$;
$\therefore \angle CEP = \angle CPE$, and $\therefore CE = CP = BD$

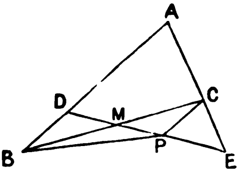
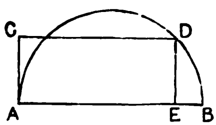


FIG. 345.

II. 14. (1) Let AB be the sum of the sides.
On AB describe a semicircle ADB.
Draw $AC \perp$ to AB and equal to a side of the given square.
Draw $CD \parallel$ to AB, and $DE \perp$ to AB.
Then rect. AE, EB = sq. on ED,
= sq. on CA,
and sum of $AE, EB = AB$.



(2) Let AB be the difference of the sides.
On AB as diameter describe a $\odot AEBD$.
Draw $AC \perp$ to AB and equal to a side of the given square.
Draw $CEOD$ through O the centre.
Then rect. CD, CE = sq. on AC;
and difference of $CD, CE = AB$.

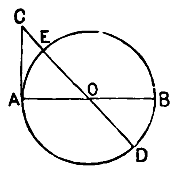


FIG. 346.

III. 36. Join AO, and produce it to meet DE in F.
Then since rect. AC, AE = rect. AB, AD,
a $\odot$ may be described about $CBDE$;
$\therefore \angle ABC = \angle AEF$.
Join CQ. Then $\because \angle ABC = \angle AQC$,
$\therefore \angle AEF = \angle AQC$;
$\therefore$ a $\odot$ may be described about $CEFQ$;
$\therefore \angle QCE + \angle QFE$ = two rt. $\angle$s;
and $\angle QCE$ is a rt. $\angle$, since AQ is a diameter;
$\therefore \angle QFE$ is a rt. $\angle$.

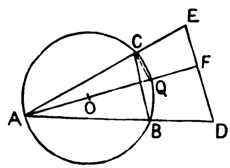


FIG. 347.

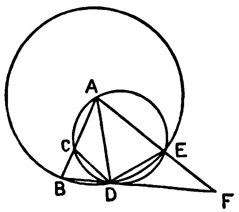


FIG. 348.

IV. 10. Take the diagram in Euclid I
10, and complete it as directed.

Then $\angle AED$ = supplement of $\angle ACD$,
$= \angle BCD = \angle ABD$;
$\therefore \angle ADE = \angle AED = \angle ABD = \angle ADB$
$\therefore$ third $\angle EAD$ = third $\angle BAD$.
Then $\angle BAF = 2 \angle BAD = \angle ABD$;
and $\angle ABF = 2 \angle BAD = \angle ADB$;
$\therefore \angle AFB = \angle BAD$.

1861. I. 32. Let EF be the given st. line.
Draw $FP \parallel$ to CB, making an acute $\angle$ with EF.
Make $\angle NCB = \angle EFP$, CN meeting AB in N.
Draw $QNM \parallel$ to EF.
Then $\because \angle NCM = \angle EFP = \angle NMC$,
$\therefore CN = NM$

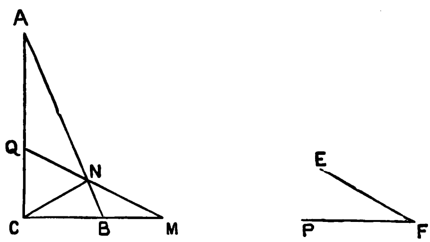


FIG. 349.

Also, $\angle QCN$ = complement of $\angle NCM$,
= complement of $\angle NMC$,
$= \angle CQN$,
$\therefore QN = NC = NM$.

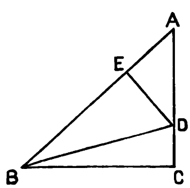


FIG. 350.

II. 13. Sq. on BD = sqq. on BA, AD diminished by 2 rect. BA, AE;

and sq. on BD = sqq. on BA, AD diminished by 2 rect. AD, AC;
$\therefore$ 2 rect. BA, AE = 2 rect. AD, AC;
$\therefore$ rect. BA, AE = rect. AD, AC.

III. 32. Let the tangents meet in T.

$$\begin{aligned} \angle TFE &= \angle FCA, && \text{(III. 22.)} \\ &= \text{supplement of } \angle FBA, \\ &= \text{complement of } \angle CBF, \\ &= \text{complement of } \angle CAF. \end{aligned}$$

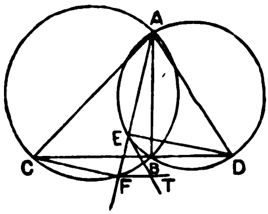


FIG. 351.

$$\begin{aligned} \angle TEF &= \angle EDA, && \text{(III. 22.)} \\ &= \text{complement of } (\angle EDB + \angle DAB), \\ &= \text{complement of } (\angle EAB + \angle DAB), \\ &= \text{complement of } \angle DAF, \\ &= \text{complement of } \angle CAF, \end{aligned}$$

$\therefore \angle TFE = \angle TEF$;

$\therefore TE = TF$;

and these tangents being equal, T must be on the common chord of the $\odot$s, *i.e.* on AB produced. (See p. 171, Ex. 23.)

The proof is similar when FEA bisects the exterior angle between CA and DA.

IV. 4. The first part of this rider has been proved in Ex. 4 to IV. 4.

Let the inscribed $\odot$ touch BC in D, and the escribed $\odot$ touch BC in E.

Then, from the properties of tangents,

$2AB + 2CD =$ sum of sides of $\triangle$;

$2AB + 2BE =$ sum of sides of $\triangle$;

$\therefore CD = BE$;

$$\begin{aligned} \therefore ED &= BD - BE, \\ &= BD - DC, \\ &= BF - CH, \\ &= AB - AC. \end{aligned}$$

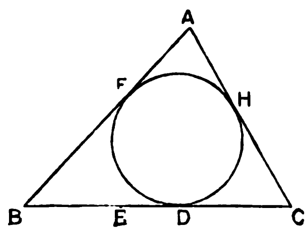


FIG. 852.

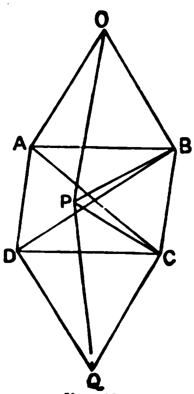


FIG. 353.

1862. I. 4. Let P, O, Q be the vertices of equilateral $\triangle$s.

Then $\angle PCQ = \angle PCD + \angle QCD,$
$= \angle PCD + \angle PCB,$
$= \angle BCD;$
also $PC = BC$, and $CQ = CD$;
$\therefore PQ = DB.$ (I.

Again, $\angle OBP = \angle OBA + \angle ABP,$
$= \angle PBC + \angle ABP,$
$= \angle ABC;$
and $OB = AB$, and $BP = BC$;
$\therefore OP = AC.$ (I.

II. 10. Let AB be the given line.

Take AE, in AB produced, such that sq. on $AE = 3$ sq. on A cut off $AC = BE$.

FIG. 354.

Then $CE = AB$.
Make $CD = AC$; then $CB = D$
Now since AD is bisected i
and produced to E,

sum of sqq. on AE, DE = twice sum of sqq. on AC, CE;
$\therefore$ 3 sq. on AB + sq. on CB = 2 sq. on AC + 2 sq. on AB;
$\therefore$ sq. on AB + sq. on CB = 2 sq. on AC.

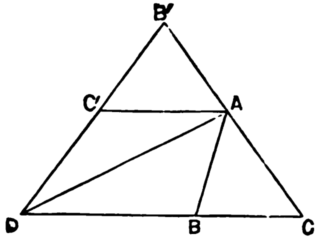


FIG. 355.

III. 28. Let $AB'C'$ be the position of the $\triangle ABC$ when it been turned about A.

Let the two positions of the produced intersect in D. Join A

Then

$\angle AC'D$ = supplement of $\angle AC$
= supplement of $\angle AC$.

$\therefore$ a circle can be described ab $ACDC'$; and since chord AC = ch AC',

$\therefore \angle ADC' = \angle ADC.$

IV. 10. In arc AED of the smaller ⊙ take any pt. E.

Then $\angle AED$ = supplement of $\angle ACD$,
$= \angle BCD$,
$= \angle ABD$;

∴ chord AD subtends equal angles in the circle AED and in the circle described about $\triangle ABD$;

∴ these ⊙s are equal.

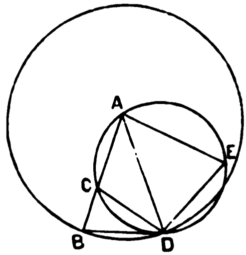


FIG. 356.

1863. I. 47. Draw the ⊥s from A, B, C to opposite sides meeting those sides in α, β, γ; α_1, β_1, γ_1.

Then sqq. on $B\alpha$, $A\alpha$, $C\beta$, $B\beta$, $A\gamma$, $C\gamma$,
= sqq. on AB, BC, CA,
= sqq. on $C\alpha$, $A\alpha$, $B\beta$, $A\beta$, $B\gamma$, $C\gamma$.

And taking away common squares,
sqq. on $B\alpha$, $C\beta$, $A\gamma$ = sqq. on $C\alpha$, $A\beta$, $B\gamma$;
∴ sqq. on $B_1\alpha_1$, $A\alpha_1$; $C_2\beta_1$, $B\beta_1$; $A_3\gamma_1$, $C\gamma$,
= sqq. on $C_1\alpha_1$, $A\alpha_1$; $A_2\beta_1$, $B\beta_1$; $B_3\gamma_1$, $C\gamma_1$;
∴ sqq. on AB_1, BC_2, CA_3 = sqq. on AC_1, BA_2, CB_3.

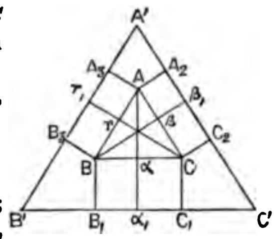


FIG. 357.

II. 11. Let AB be the line to be divided.

On AB describe a square $ABDC$, produce CA to E so that AE = the other given line.

Complete rectangle $AEMB$, and make rectangle $ECKG = AEMB$. Let GK cut AB in H.

Then, taking away rectangle EH,
$AK = GB$;
or, rect. AB, AH = rect. HB, AE;
∴ H is the point required.

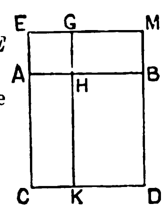


FIG. 358.

III. 28. Solved before as Exercise 31, on p. 172.

1864. I. 38. Let $ABCD$ be the quadrilateral, having $AB \parallel$ to
Join AC, BD, intersecting in O; bisect AB in E; join EO. T
EO produced shall pass through the middle pt. of DC.

For, if not, let F be the middle pt. of DC, such that EPF is a
line, and join FO.

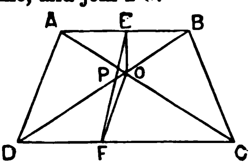


FIG. 359.

Then $\triangle DAB = \triangle CAB$,
and $\triangle OAB$ is common,
$\therefore \triangle DAO = \triangle CBO$.
Also $\triangle EAO = \triangle EBO$,
and $\triangle FDO = \triangle FCO$;
$\therefore$ figure $DAEOF$=figure $CBEOF$
$\therefore$ figure $DAEOF$=half of figure AB

But $\triangle FEA = \triangle FEB$, and $\triangle AFD = \triangle BFC$;
$\therefore$ figure $DAEPF$=figure $CBEPF$;
$\therefore$ figure $DAEPF$=half of the figure $ABCD$,
=figure $DAEOF$, which is absurd.

Hence F is not the middle pt. of DC.

Similarly, it may be shown that no point, except one in EO
duced, is the middle pt. of DC.

II. 14. Bisect the given st. line AB in P, and divide it in
that rect. AB, BQ=given rectilineal figure. (I. 45 and II.

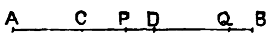


FIG. 360.

Take $BD=PQ$, and divide AD
(if possible), so that rect. AC, CD=g
rectilineal figure (II. 14); then C a
shall be the points of section require

For rect. AC, CD=given figure=rect. AB, BQ=rect. AB, D.
And sq. on AD=sqq. on AC, CD with 2 rect. AC, CD,
=sqq. on AC, CD with 2 rect. AB, DP,
=sqq. on AC, CD with 4 rect. AP, DP;
also, sq. on AD=sqq. on AP, PD with 2 rect. AP, DP,
=sqq. on BP, PD with 2 rect. AP, DP,
=sq. on BD with 4 rect. AP, DP; (*Eucl.* II
$\therefore$ sqq. on AC, CD=sq. on BD.

III. 36. Let C be the centre of the circle, $ABCD$ the diameter. It is evident that PS, QR, intersect on some point O in AD.

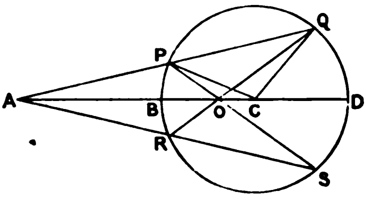


FIG. 361.

Also $\angle PCA = \frac{1}{2} \angle PCR = \angle PQR$;

$\therefore$ $POCQ$ can be circumscribed by a $\odot$;

$\therefore$ rect. AO, AC = rect. AP, AQ,

= rect. AB, AD;

$\therefore$ O is a fixed point.

IV. 11. Let Z, A, B, C . . . be consecutive angles of the figure.

Join ZA, AB, BC, CD, AC, BD.

Then since $\angle ABC = \angle BCD$,

$\therefore$ arc AC = arc BD;

and, taking away the common part BC,

arc AB = arc CD;

and $\therefore$ side AB = side CD.

Similarly, side $AB = CD = EF = \ldots = ZA = BC = \ldots$ the number of sides being *odd*. So all the sides are equal.

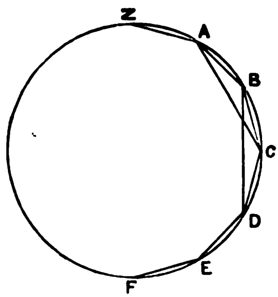


FIG. 362.

1865. I. 20. Let AB be the given line, P and Q the given points. Join PQ, and produce it to meet AB, or AB produced, in R. Then shall PQ be greater than the difference of any two lines drawn from P and Q to the same point in AB, as PD, QD.

For sum of PQ, QD is greater than PD;

$\therefore$ PQ is greater than the difference between PD and QD.

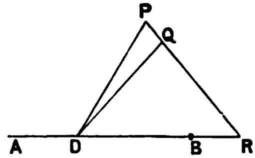


FIG. 363.

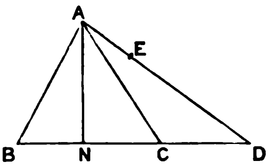


Fig. 364.

II. 12. Draw $AN \perp$ to, and ∴ bisecting, BC.

Then sq. on AD = sqq. on AC, with 2 rect. CD, CN;

∴ rect. AD, AE with rect. AD, = sqq. on AC, CD with 2 rect. CN;

∴ rect. AD, DE = sq. on CD with rect. CD, BC,
= rect. BD, CD. (II.

III. 18. Let DC be $\perp$ to AB the line joining the centres A an
Draw DP, DQ, tangents to the ⊙s. Join AP, AD, BQ, BD.

Then sq. on PD = sq. on AD − sq. on AP,
sq. on QD = sq. on BD − sq. on BQ;

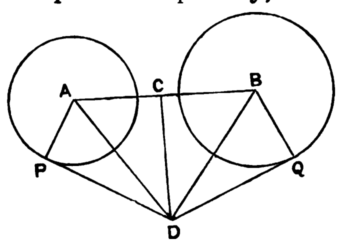


Fig. 365.

∴ difference of sqq. on PD, QD = difference of sqq. on AD, B
with difference of sqq. on BQ, AP.

Now difference of sqq. on AD, BD = difference of sqq. on AC, B
= a constant;

and difference of sqq. on BQ, AP = a constant;

∴ difference of sqq. on PD, QD is constant.

IV. 5. Let AB be the given side. Then since the centre of the circumscribed ⊙ is fixed, the angle ACB is of constant magnitude.

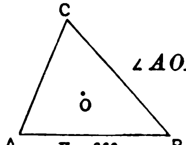


Fig. 366.

∴ sum of ∠s CAB, CBA is constant;

and, if O be the centre of the inscribed ⊙

∠ AOB = supplement of half the sum of ∠s CAB, CBA;

∴ ∠ AOB is constant, and the locus of O is therefore a ⊙ passing through A and B.

1866. I. 33. Let $AB=CD$, and $\angle CAB = \angle BDC$.

Then since sides AB, BC in $\triangle ABC$ are equal to sides BC, CD in the other, and $\angle$s ACB, CBD are both acute, it may be shown as in I. Prop. E, p. 43, that $AC=BD$, and $\angle ABC = \angle BCD$, and $\angle ACB = \angle CBD$, and $\therefore ABDC$ is a $\square$.

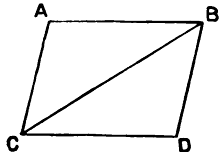


FIG. 367.

But if $\angle$s at A and D be acute, $\angle$s ACB, CBD are not necessarily *both* acute or *both* obtuse, and $\therefore$ I. Prop. E will not apply.

II. 9. We have to prove that

$$Ag+DE=2\,(af+Cd),$$
$$\text{or, } 2Ac+DE=af+Cd,$$
$$\text{or, } Ac+\triangle DBE=\triangle eac+\triangle cCD.$$

Draw $ckK \perp$ to ED.

Then $Ac-\triangle cCD+\triangle DBE$

$$=Ac-\triangle cCk+\triangle AkK,$$
$$=\triangle aKc,$$
$$=\triangle eac.$$

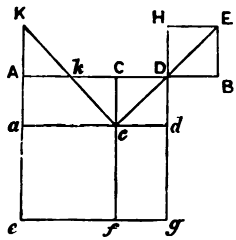


FIG. 368.

III. 32. Let AT, AT' be tangents to the $\odot$s passing through ABD, ABC.

Then $\angle TAD = \angle ABD$ in alternate segment,

and $\angle T'AC = \angle ABC$ in alternate segment,

$\therefore \angle T'AT = \angle CAD + \angle ABD + \angle ABC$,

$= \angle CAD + \angle CBD$.

So angle between tangents to the $\odot$s passing through ACD, CBD,

$= \angle ACB + \angle ADB$,

$=$four rt. $\angle$s diminished by sum of $\angle$s CAD, CBD.

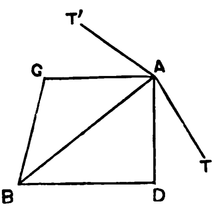


FIG. 369.

Hence the acute angles between them are the same in both cases.

IV. 2. As the third side is at a given distance from the centre, it is of a given length, and therefore subtends a given angle at the circumference. On the line joining the two given points describe the segment of a circle capable of containing the given angle. The intersection of this segment with the given circle will give the vertex of the triangle.

Note.—There will generally be two solutions, or none at all; but in the particular case when the segment touches the circle, only one solution.

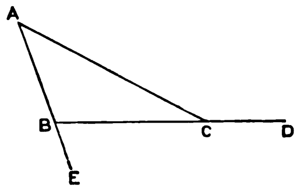


FIG. 370

1867. I. 16. Let CBE, ACD any two exterior $\angle$s of the $\triangle ABC$.

Then $\because \angle ACD$ is greater th $\angle ABC$,

sum of $\angle$s ACD, CBE is great than sum of $\angle$s CBE, ABC;

$\therefore$ sum of $\angle$s ACD, CBE is great than two rt. $\angle$s.

I. 43. The greatest value which each complement can have is one fourth of the parallelogram, when $AE=ED$.

II. 11. Solved before. See Riders in 1862.

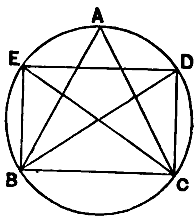


FIG. 371 (1).

III. 22. Let DB, EC be equal chords bisectin two $\angle$s in the $\triangle ABC$ inscribed in a $\odot$.

Now $\angle DCB = \angle EBC$, subtended by equa chords;

$\therefore$ sum of $\angle$s EDC, DCB=two rt. $\angle$s;

$\therefore ED$ is $\parallel$ to BC;

$\therefore \angle DBC = \angle EDB$,

$= \angle ECB$;

and $\therefore \angle ABC = 2\angle DBC = 2\angle ECB = \angle ACB$

Next, if the bisectors be on opposite sides of the centre,

$\angle EBC = \angle BAD$, subtended by equal chords.

But $\angle EBC = \angle ABC + \angle EBA$,

$= \angle ABC + \angle ECA$,

$= \angle ABC + \frac{1}{2}\angle ACB$;

and $\angle BAD = \angle BAC + \angle CAD$,

$= \angle BAC + \angle CBD$,

$= \angle BAC + \frac{1}{2}\angle ABC$;

$\therefore \angle BAC + \frac{1}{2}\angle ABC = \angle ABC + \frac{1}{2}\angle ACB$;

$\therefore \angle BAC = \frac{1}{2}(\angle ABC + \angle ACB)$,

and $\therefore \angle BAC$=two-thirds of a rt. $\angle$.

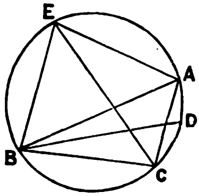


FIG. 371 (2).

1868. I. 41. Through Q draw $RQS \parallel$ to NM.

Then $\triangle BQD = \triangle ODQ + \triangle OQB$,
$= \frac{1}{2} \square NQ + \frac{1}{2} \square OS$,
$= \frac{1}{2} \square NRSM$.

And $\triangle PQD = \triangle PDC - \triangle QDC$,
$= \frac{1}{2} \square NC - \frac{1}{2} \square RC$,
$= \frac{1}{2} \square NRSM$.

$\therefore \triangle BQD = \triangle PQD$;

$\therefore PB$ is $\parallel$ to QD.

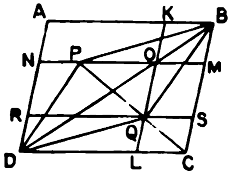


FIG. 372.

II. 12. Sq. on AC=sqq. on AD, DC + 2 rect. CD, DK.
And sqq. on AB, AC = 2 sqq. on AD, DC. (II. 13, Ex.)

$\therefore$ 2 sq. on AC=sqq. on AB, AC + 2 rect. BC, DK;
$\therefore$ sq. on AC=sq. on AB + 2 rect. BC, DK;
and sq. on BC=sq. on AB + 2 rect. AC, EL;
and sq. on BC=sq. on AC - 2 rect. AB, FM;

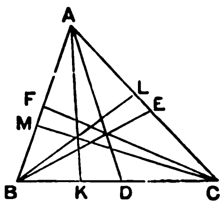


FIG. 373.

$\therefore$ 2 rect. BC, DK=sq. on AC - sq. on AB;
2 rect. AB, FM=sq. on AC - sq. on BC;
2 rect. AC, EL=sq. on BC - sq. on AB;
$\therefore$ 2 rect. BC, DK = 2 rect. AB, FM + 2 rect. AC, EL;
$\therefore$ rect. BC, DK=rect. AB, FM + rect. AC, EL.

III. 35. Let P be the point, C the centre, ACA' a diameter through P, NM the side of the given square, and describe a semicircle on NM, and place NO in it, NO being the side of a square=sq. on CA - sq. on CP.

With centre P and radius OM, describe a ⊙ cutting the original ⊙ in Q.

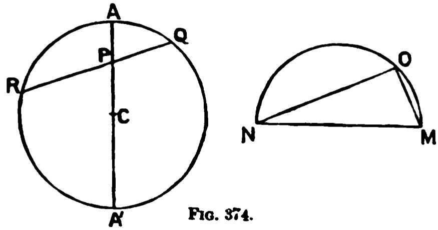


FIG. 374.

Then rect. RQ, PQ = rect. RP, PQ + sq. on PQ,
= rect. AP, PA' + sq. on PQ,
= sq. on CA − sq. on CP + sq. on PQ,
= sq. on ON + sq. on OM,
= sq. on NM.

The limits are deducible from the fact that OM must be greater than PA and less than PA'.

IV. 4. Let O be the centre of the ⊙ inscribed in △ ABC; then OA, OB, OC bisect the angles at A, B, C.

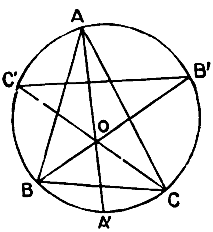


FIG. 375.

$\therefore \angle AOC' = \angle OAC + \angle OCA,$
$= \frac{1}{2} \angle BAC + \frac{1}{2} \angle ACB;$
and $\angle B'C'O = \angle B'BC,$
$= \frac{1}{2} \angle ABC;$
$\therefore \angle AOC' + \angle B'C'O$
$= \frac{1}{2}(\angle BAC + \angle ACB + \angle ABC),$
= a rt. angle;
$\therefore AA'$ is $\perp$ to $B'C'$;
$\therefore O$ is the centre of $\perp$s of △ $A'B'C'$.

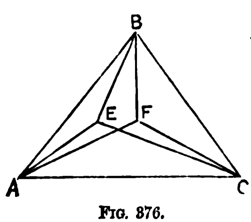


FIG. 376.

1869. I. 40. Subtracting the sums of △s AEB, BEC, and AFB, BFC from the whole △ ABC,

△ AEC = △ AFC;
$\therefore EF$ is ‖ to AC.

If the points E, F, or either, lie on the opposite side of AB, the △s AEB, AFB are (both or one) negative, and similarly for the other sides of △ ABC.

II. 11. In the diagram of the Proposition

$EF = EB$, and $\therefore$ EF is less than $EA + AB$;

$\therefore$ AF is less than AB;

$\therefore$ AH is less than AB;

$\therefore$ H lies between A and B.

III. 33. Let ABC be the $\triangle$ inscribed in the $\odot$ PQR.
Let the segments APC, ARB, when folded meet in O.
Then $\angle AOC = \angle$ in segment APC,
$=$supplement of $\angle ABC$;
so $\angle AOB =$supplement of $\angle ACB$;
$\therefore$ $\angle BOC =$supplement of $\angle BAC$,
$= \angle$ in segment BQC;
so that when BQC is folded it must pass through O.

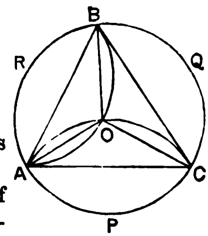


FIG. 377.

Should one $\angle$ be obtuse, the circumferences of two of the segments when produced will intersect on the third.

IV. 11. Let O be the centre of the regular pentagon $ABCDE$.

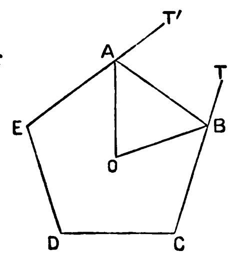


FIG. 378.

Produce CB, EA to T, T'.
Then $\angle TBA = \angle T'AB = \frac{1}{5}$ of 4 rt. $\angle$s,
$= \angle AOB$;
$\therefore$ the circles all pass through O.

1870. I. 26. Take in AB a part $BN = BE$,
and $\therefore$ $AN = EC$.
Then $\angle FEC + \angle AEB =$a rt. $\angle$,
$= \angle AEB + \angle BAE$;
$\therefore$ $\angle FEC = \angle NAE$.
Also $\angle ANE = \angle NBE + \angle NEB$,
$=\frac{3}{2}$ of a rt. $\angle$,
$= \angle ECF$.
Now since $\angle FEC = \angle NAE$,
and $\angle ECF = \angle ANE$, and $AN = EC$,
$\therefore$ $FE = AE$.

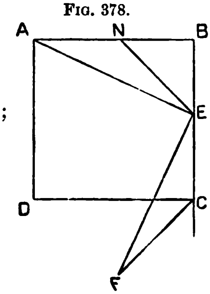
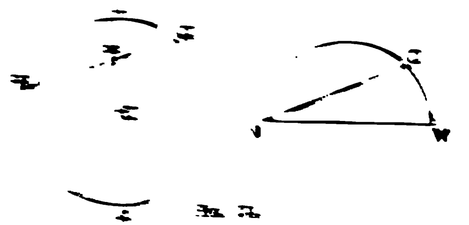


FIG. 379.

With centre ... and radius ... describe a ⊙ cutting the original ...



... then

$\angle ... = \frac{1}{2}\angle BAC - \frac{1}{2}\angle ACB$

and ... $= \frac{1}{2}\angle ABC$;

$= \frac{1}{2}(\angle BAC + \angle ACB + ...)$

= a rt. angle;

∴ AA' is ⊥ to BC;

∴ O is the centre of ...

Fig. 251.

1869. I. 40. ...

AEB, BDC, and ...

$\angle ABC$.

II. 9. By the Proposition

sqq. on DE, $EB = 2$ sqq. on DO, OE;
sqq. on AE, $EC = 2$ sqq. on AP, EP.

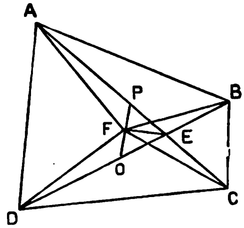


FIG. 380.

$\therefore$ sum of sqq. from E to the angular pts.,
$= 2$ sq. on $DO + 2$ sq. on $AP + 2$ (sq. on $OE +$ sq. on EP),
$= 2$ sq. on $DO + 2$ sq. on $AP + 4$ (sq. on $EF +$ sq. on FP),
(By Ex. on p. 91.)
$= 2$ sq. on $DO + 2$ sq. on $AP + 4$ sq. on $EF + 2$ sq. on $FP + 2$ sq. on FO,
$= 2$ (sq. on $DO +$ sq. on FO) $+ 2$ (sq. on $AP +$ sq. on FP) $+ 4$ sq. on EF,
$=$ sum of sqq. on FD, $FB +$ sum of sqq. on FA, $FC + 4$ sq. on EF,
$=$ sum of sqq. from F to the angular pts. $+ 4$ sq. on EF.

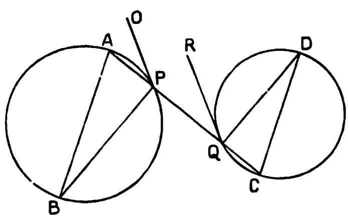


FIG. 381.

III. 32. Let OP be the tangent at P, and RQ the tangent at Q. Then

$\angle OPA = \angle ABP$,
$=$ complement of $\angle PAB$,
$=$ complement of $\angle QCD$,
$= \angle QDC$,
$= \angle RQP$;
$\therefore$ OP is $\parallel$ to RQ.

IV. 10. Each of the angles at the base of the triangle ABD in the diagram of the Proposition is $\frac{2}{5}$ of two rt. angles.

From the centre of the given circle draw five radii inclined to each other successively at angles $= \frac{2}{5}$ of two rt. angles.

The tangents at the extremities of these radii will form the circumscribing regular pentagon.

1871. I. 38. $\triangle CA'B' = \triangle AA'C + \triangle AA'B'$,
$= \triangle AA'C + \triangle ABA'$,
$= \triangle ABC$.
$\triangle BC'A' = \triangle ABA' + \triangle AA'C'$,
$= \triangle ABA' + \triangle AA'C$,
$= \triangle ABC$.
$\triangle AB'C' = \triangle BB'C' - \triangle ABB'$,
$= \triangle BB'C - \triangle ABB'$,
$= \triangle ABC$.

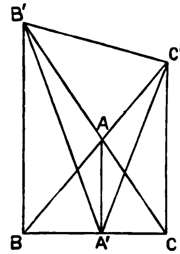


FIG. 382.

II. 10. Let AC be the given st. line.
Produce it to B, so that $CB = AC$.
Draw $CE \perp$ to AB, and $= AC$.
Join AE, EB and produce AB to D, so that $BD = EB$.
Then sq. on AD + sq. on $BD = 2$ sq. on AC + 2 sq. on CD.
But sq. on BD = sq. on EB.
$= 2$ sq. on AC.
$\therefore$ sq. on $AD = 2$ sq. on CD.

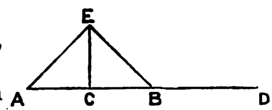


FIG. 383.

III. 32. Let $ABCD$ be the quadrilateral.
Let QT, QT' be the tangents at Q.
Then $\angle TQB = \angle PCQ$,
$= \angle DAB$;
$\therefore QT$ is $\parallel$ to PD, and so also QT' is $\parallel$ to CP.
Again, let PR, PR' be the tangents at P.
Then PR', PR are $\parallel$ to QD and QA respectively.
$\therefore$ the new figure formed is placed precisely similar with respect to QP as the old one ($ABCD$) is with respect to PQ, and the figures are equal in all respects; and any line joining similar points with respect to the figures must bisect PQ, and $\therefore$ in particular the line joining the *centres* spoken of must bisect PQ.

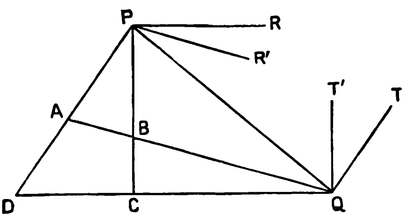


FIG. 384.

IV. 5. Let ABC be the $\triangle$; O the centre of $\perp$ s.
Then since $\angle$s at c and b are rt. $\angle$s,

$\therefore \angle\, cOb$ is supplement of $\angle\, cAb$,

$\therefore \angle\, BOC$ is supplement of $\angle\, cAb$.

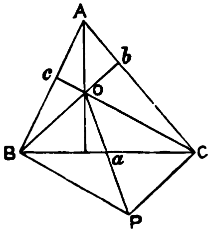


FIG. 385.

Let a be the middle pt. of BC, and pı Oa to P, so that $aP = aO$.

Then $BOCP$ is a parallelogram,

and $\angle\, BPC = \angle\, BOC$,

$=$ supplement of $\angle$

$\therefore P$ lies on the Oce of the $\odot$ about and since O is fixed, a must lie on the O $\odot$ of half the radius of the $\odot$ circums ABC, O being their centre of similitude.

1872. I. 47. Draw AP, AQ $\perp$s to EB, CD.
Then since $\triangle$s EAB, CAD are equal, and their bases EB, (equal,

$\therefore$ their altitudes AP, AQ are equal.

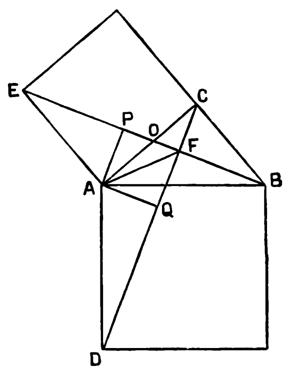


FIG. 386.

Again $\angle\, AEO = \angle\, FCO$, and $\angle\, EOA = \angle\, COF$,

$\therefore \angle\, OFC = \angle\, OAE =$ a rt. angle;

$\therefore PAQF$ is a square;

and $\therefore AF$ bisects $\angle$ EFD.

III. 22. Through B draw RBD, meeting the ⊙s in R, D.

Then $\angle P'DB$ = supplement of $\angle BAP'$,

$= \angle BAP$,

$= \angle Q'AB$;

$\therefore P'B = Q'B$.

Similarly, $PB = QB$.

Also, $\angle PBQ = \angle PAQ$,

$= \angle P'AQ'$,

$= \angle P'BQ'$;

and $\therefore \angle PBP' = \angle QBQ'$.

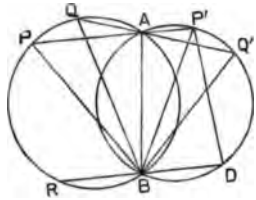


Fig. 387.

Hence in △s PBP', QBQ',

$\because PB = QB$, and $P'B = Q'B$, and $\angle PBP' = \angle QBQ'$,

$\therefore PP' = QQ'$.

IV. 4. Let A be the given angular point, E the centre of ⊙ inscribed n the △.

Join AE and produce it to cut the circumscribing ⊙ in D.

With D as centre, and DE as radius, describe a ⊙ cutting the given ⊙ in B and C.

Then shall ABC be the required △.

For DA bisected $\angle BAC$, since $DB = DC$;

$\therefore$ if EC bisects $\angle ACB$, E will be the centre f the ⊙ inscribed in △ ABC.

Now $\angle DCB = \angle DAB$,

$= \frac{1}{2} \angle BAC$.

Also in the isosceles △ EDC,

um of $\angle$s DEC, ECD = supplement of $\angle EDC$,

= supplement of $\angle ABC$,

= sum of $\angle$s BAC, ACB;

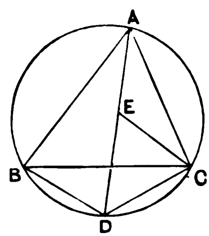


Fig. 388.

$\therefore \angle ECD = \frac{1}{2}$ sum of $\angle$s BAC, ACB;

and $\angle DCB = \frac{1}{2} \angle BAC$,

$\therefore \angle ECB = \frac{1}{2} \angle ACB$.

BOOK VI.

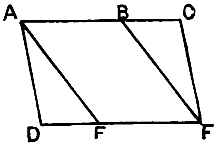


FIG. 389.

Page 245.

EXERCISE 1. Since the altitudes of the $\triangle$s *ADE*, *FCB* are equal,

$$\therefore \triangle ADE : \triangle FBC = DE : BC.$$

Ex. 2. Let P be a point in BD, a diagonal of the $\square$ $ABCD$, and join PA, PC.

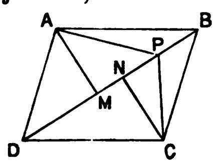


FIG. 390.

Draw AM, CN $\perp$s to BD.

Then $\because \angle ADM = \angle NBC$,

and $\angle AMD = \angle BNC$, and $AD = BC$,

$$\therefore AM = CN.$$

Since then the altitudes of $\triangle$s APD, CPD on the same base are equal,

$$\therefore \triangle APD = \triangle CPD;$$

and for the same reason $\triangle APB = \triangle CPB$.

Page 246.

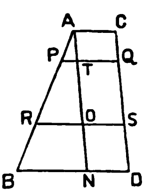


FIG. 391.

EXERCISE 1. Let AB, CD be any two straight lines cut by the parallels PQ, RS, BD.

Draw $AN \parallel$ to CD, meeting BD in N.

Then $BR : RA = NO : OA$, (vi. 2.)

and $\therefore BR : NO = RA : OA$; (v. 15.)

and, similarly, $RP : OT = RA : OA$;

$\therefore BR : NO = RP : OT$ (v. 5.)

$\therefore BR : RP = NO : OT$ (v. 15.)

$= DS : SQ$. (I. 34.)

Ex. 2. Let AB be $\parallel$ to CD.
Draw $AO \parallel$ to BD, cutting PQ in N.
Then $CP : PA = ON : NA$,
$= DQ : QB$.

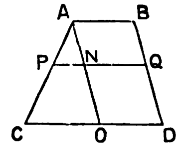


FIG. 392.

Ex. 3. Let AB, CD, EF, GH be four parallel straight lines.
Then $AE : EO = BF : FO$;
$\therefore AE : BF = EO : FO$,
$= CO : DO$,
$= CG : DH$.

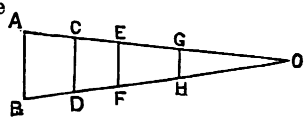


FIG. 393.

Ex. 4. Let $ABCD$ be the quadrilateral; M, N, O, P the middle pts. of the sides. Join AC.
Then $\because AM : MB = CN : NB$;
$\therefore AC$ is $\parallel$ to MN;
and $\because AP : PD = CO : OD$;
$\therefore AC$ is $\parallel$ to PO.
Hence PO is $\parallel$ to MN, and similarly, if BD be joined, we can show that PM is $\parallel$ to ON;
$\therefore MPON$ is a $\square$.

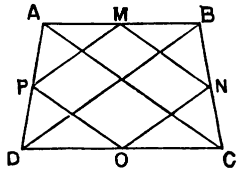


FIG. 394.

Ex. 5. Let $ABCD$ be a trapezium, having $AD \parallel$ to BC.
Produce BA, CD to meet in E.
Draw EN to the intersection of the diagonals, and let it meet the $\parallel$ sides in O, M.
Then $\because \triangle BAC = \triangle CDB$, $\therefore \triangle BNA = \triangle DNC$;
also $\triangle BAN : \triangle BNE = BA : BE$,
$= CD : CE$,
$= \triangle DNC : \triangle CNE$;
$\therefore \triangle BNE = \triangle CNE$.
Again, $\triangle ECN : \triangle MCN = EN : NM$,
$= \triangle BNE : \triangle BNM$;
$\therefore \triangle BNM = \triangle MCN$, and $\therefore BM = CM$.
Now $BM : ME = AO : OE$;
and $CM : ME = DO : OE$;
$\therefore AO = DO$.

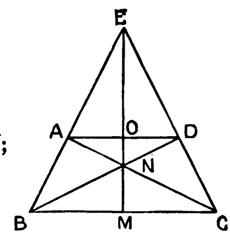


FIG. 395.

Page 248.

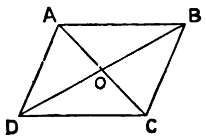


Fig. 396.

Exercise 1. Let $ABCD$ be a ▱, and let O be the pt. where the diagonals bisect each other.

Then if OB bisects ABC,

$$AB : BC = AO : OC;$$

and $\therefore AB = BC$; but not otherwise.

Ex. 2. Let BC be the given straight line.

On BC describe the equilateral $\triangle\, BAC$.

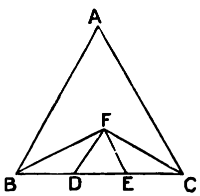


Fig. 397.

Bisect $\angle\, ABC$ and $\angle\, ACB$ by st. lines, meeting in F.

Draw FD, FE $\parallel$ to AB, AC respectively.

Then $\because \angle\, DFB = \angle\, FBA = \angle\, FBD$;

$\therefore FD = BD$, and similarly $FE = EC$.

Now $\triangle\, FDE$ is equiangular to $\triangle\, ABC$, and is therefore an equilateral $\triangle$.

$$\therefore DE = FD = FE;$$
$$\therefore BD = DE = CE.$$

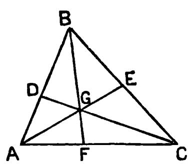


Fig. 398.

Ex. 3. Let EA, DC be bisectors of $\angle$s BAC, BCA, and let them intersect in G.

Join BG, and produce it to meet AC in F.

Then shall BF bisect $\angle\, ABC$.

For $BC : CF = BG : GF$,

$$= BA : AF;$$
$$\therefore BC : BA = CF : AF;$$

and $\therefore FB$ bisects $\angle\, ABC$.

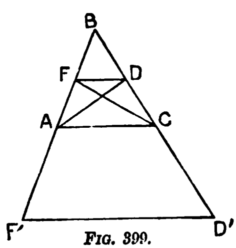


Fig. 399.

Ex. 4. Since $BF : FA = BC : CA$.

$$\therefore BF : BF + FA = BC : BC + CA;$$
$$\therefore BF : BA = BC : BD'.$$

Similarly, $BD : BC = BA : BF'$.

$$\therefore \text{rect. } BF, BD' = \text{rect. } BA, BC,$$
$$= \text{rect. } BD, BF';$$
$$\therefore BF : BD = BF' : BD';$$
$$\therefore FD \text{ is } \parallel \text{ to } F'D'.$$

Ex. 5. Let ABC be an isosceles $\triangle$. Draw CD, BE, bisecting the $\angle$s at the base. Join DE.

Then $AD : DB = AC : BC$,

$= AB : BC$,

$= AE : EC$;

$\therefore DE$ is $\parallel$ to BC.

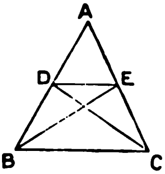


Fig. 400.

Page 250.

Exercise 1. If the angles at the base are equal, the bisector of the exterior angle at the vertex will be parallel to the base. (See p. 52, Ex. 2, and p. 345, Rider set in 1860.)

Ex. 2. Draw $BOP \perp$ to CD, and make $OP = OB$. Join PA, cutting CD in D. Join DB.

Then $\triangle$s DOP, DOB are equal in all respects;

$\therefore CD$ bisects $\angle PDB$;

$\therefore AD : DB = AC : CB$.

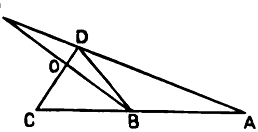


Fig. 401.

Ex. 3. Divide BC in O, so that

$BO : OC = BD : DC$.

Then $\because BO : OC = BA : AC$;

$\therefore AO$ bisects $\angle BAC$;

$\therefore \angle OAC + \angle CAD$

$= \frac{1}{2}(\angle BAC + \angle CAE)$,

$=$ a right angle.

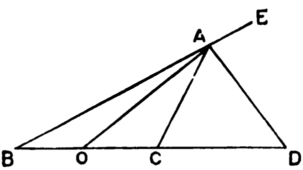


Fig. 402.

Ex. 4. Let AF, AD be the internal and external bisectors; AP the perpendicular to AB.

Then, by Ex. 3, $\angle FAD$ is a rt. angle;

$\therefore$ a circle described on FD as diameter will pass through A;

and since $\angle BAF = \angle ADF$, BA is a tangent to this $\odot$;

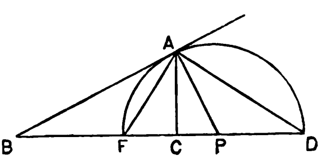


Fig. 403.

and AP being $\perp$ to AB will be a radius of the $\odot$; and $\therefore FP = DP$.

Page 252.

Let BAC be a given $\angle$, and take $AB : AC$ in the given ratio.

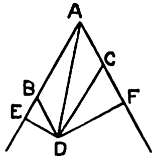


FIG. 404.

Draw $BD \parallel$ to AC, and $CD \parallel$ to AB, and join AD.
Then AD divides the $\angle\, BAC$ as is required.
Draw DE, $DF \perp$ to AB, AC, or these produced.
Then $\triangle$s BED, CFD are similar;
and $\therefore FD : ED = DC : DB$,
$= AB : AC$;
and the perpendiculars drawn from any other pt. in AD will be in the same ratio.

Page 254.

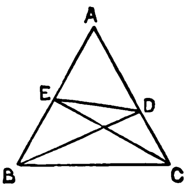


FIG. 405.

Exercise 1. Since $\triangle$s ABD, ACE are equiangular,

$\therefore BA : AD = CA : AE$;
$\therefore BA : AC = AD : AE$;

$\therefore \triangle$s AED, ABC are equiangular, by the Proposition.

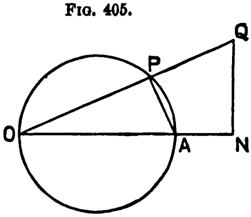


FIG. 406.

Ex. 2. Let OA be the diameter through O.
Draw $QN \perp$ to OA.
Then the $\triangle$s OPA and ONQ are similar;
$\therefore OP : OA = ON : OQ$;
$\therefore$ rect. OA, ON = rect. OP, OQ, a constant;
$\therefore N$ is a fixed point;
and the locus of Q is a straight line through N perpendicular to ON.

Miscellaneous Exercises on Propositions I. to VI.

Page 255.

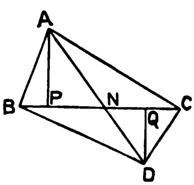


FIG. 407.

1. Let ABC, DBC be the $\triangle$s.
Join AD, and draw AP, $DQ \perp$ to BC.
Let AD cut BC in N.
Then ANP, DNQ are similar $\triangle$s;

$\therefore \triangle ABC : \triangle DBC = AP : DQ$,
$= AN : DN$.

2. On DC the radius of the $\odot$ ABC, of which AC is a diameter, describe the circle DEC.

Draw BEC cutting the $\odot$, whose diameter is DC, in E.

Then since $\angle ABC =$ a rt. $\angle = \angle DEC$;

$\therefore$ Δs ABC, DEC are similar;

$\therefore BE : EC = AD : DC$,

$\therefore BE = EC$.

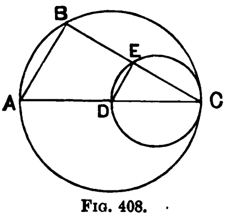


FIG. 408.

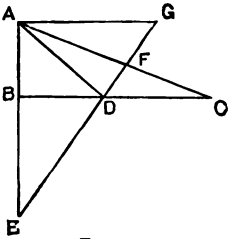


FIG. 409.

3. Since Δs AFG, CFD are similar,

$$DF : FG = DC : AG,$$
$$= BD : AG,$$
$$= ED : GE;$$
$$\therefore DE : DF = GE : FG.$$

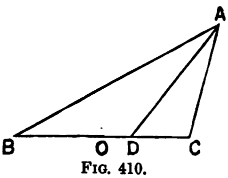


FIG. 410.

4. Since $BA : AC = BD : DC$;

$$\therefore BA - AC : BA + AC = BD - DC : BD + DC,$$
$$= 2OD : 2OB,$$
$$= OD : OB.$$

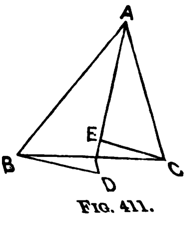


FIG. 411.

5. Let ABC be a triangle, AD the bisector of $\angle BAC$, BD, CE $\perp$s on AD from B and C.

Then $\because$ Δs ABD, ACE are similar,

$\therefore BD : AB = CE : AC$;

and $\therefore BD : CE = AB : AC$.

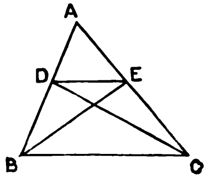


FIG. 412.

6. Take $\triangle\ ADE$ from each of the equal $\triangle$s DAC, EAB.

Then $\triangle\ DBE = \triangle\ ECD$;

$\therefore\ DE$ is $\parallel$ to BC;

$\therefore\ BD : DA = CE : EA$.

7. O is the centre of the escribed $\odot$, touching the side BC. (See IV. 4, Ex. 4.)

$\therefore\ AO$ will bisect $\angle\ BAC$;

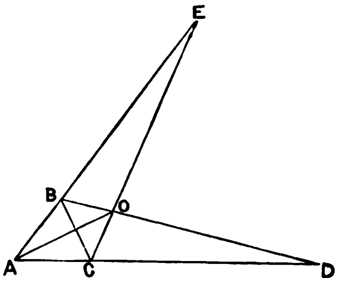


FIG. 413.

$\therefore\ AD : AB = DO : OB$;

and $OC : OE = AC : AE$.

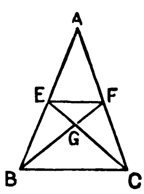


FIG. 414.

8. Since E and F are the middle pts. of AB, AC, EF is $\parallel$ to BC.

Now $\triangle\ AEC = \triangle\ BEC$;

and $\triangle\ EBC = \triangle\ FCB$; and taking $\triangle\ BGC$ from each,

$\triangle\ EBG = \triangle\ FCG$.

Hence $\triangle\ AEC - \triangle\ FCG = \triangle\ BEC - \angle\ EBG$;

$\therefore$ fig. $AEGF = \triangle\ BCG$.

9. Through O any point in BD, a diagonal of the $\square ABCD$, draw FOE, meeting AB, CD in F, E.

Then $\triangle$s BOF, DOE are similar.

$\therefore BO : OD = FO : OE.$

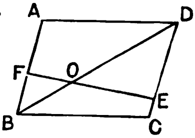


FIG. 415.

10. Since $DF : FE = BD : CE$,

$= DA : AE$;

$\therefore AF$ bisects $\angle BAC$;

$\therefore$ the locus of F is a straight line bisecting $\angle BAC$.

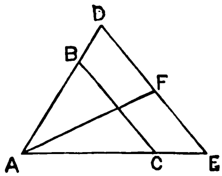


FIG. 416.

Page 258.

EXERCISE 1. Let AC be the given line. Draw AB making any angle with AC. Make AD = seven times AB.

Join CD, and draw $BE \parallel$ to CD.

Then $AE : AC = AB : AD$;

$= 1 : 7.$

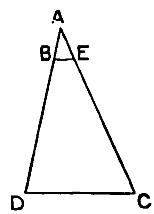


FIG. 417.

Ex. 2. Let AC be the given line.

Draw AB making any angle with AC.

Make AD = five times AB, and in BD take $BE = AB$. Join CD, and draw $EF \parallel$ to CD.

Then $AF : AC = AE : AD$;

$= 2 : 5.$

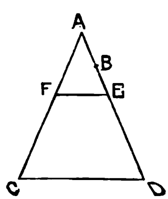


FIG. 418.

Page 259.

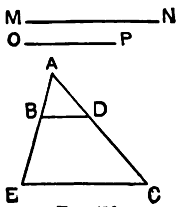


FIG. 419.

EXERCISE 1. Let MN, OP be two lines in the given ratio.

Let AB be the given st. line.

Draw $AC=MN$, making any $\angle$ with AB.

In AC take $CD=OP$.

Join BD and draw CE, meeting AB produced in E, $\parallel$ to BD.

Then $AE:EB=AC:CD$,

$=MN:OP$.

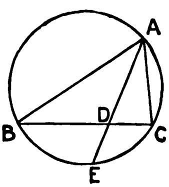


FIG. 420.

Ex. 2. Let BC be the given base. Describe on BC a segment of a $\odot$ BAC, capable of containing the given vertical $\angle$. Bisect the remaining segment BEC in E. Divide BC in D in the given ratio of the sides. Join ED, and produce it to meet the Oce in A. Then BAC is the triangle required.

For since arc $BE=$ arc CE, $\therefore$ $\angle BAD = \angle CAD$.

$\therefore BA : AC=BD:DC$.

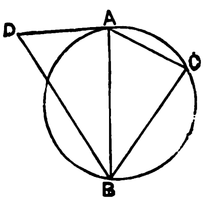


FIG. 421.

Page 261.

EXERCISE. Since $\angle DAB = \angle ACB$, in alternate segment, and $\angle ABD = \angle ABC$,

$\therefore$ $\triangle$s BAD, ABC are similar;

and $\therefore BD:DA=AB:AC$.

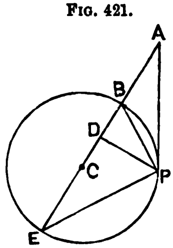


FIG. 422.

Page 262.

EXERCISE. Produce ABC to meet the Oce in E, and join EP.

Then $\angle APB = \angle BEP$ in alternate segment; and since $\triangle BPE$ is right-angled at P,

$\therefore \angle BPD = \angle BEP$; (VI. 12.)

$\therefore \angle APB = \angle DPB$.

Page 263.

EXERCISE 1. Let AB be the given st. line.

On AB as diameter describe a ⊙.

Draw $BC \perp$ to AB and $= AB$.

Draw $CDOE$ through O the centre.

Produce AB to F, so that $FB = CD$.

Then rect. $AF, FB =$ rect. EC, CD,
$= $ sq. on BC,
$= $ sq. on AB;

$\therefore$ AB is a mean proportional between AF and FB.

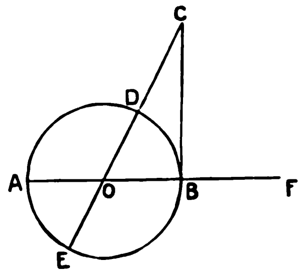


FIG. 423.

Ex. 2. Let ABC be an isosceles △.

Draw $AD \perp$ to AB, meeting BC (or BC produced) in D. Draw AE to E the middle pt. of BD.

Then $\because$ BAD is a rt. ∠, $\therefore$ E is centre of ⊙ described about △ ABD.

$\therefore \angle EAB = \angle ABE = \angle ACB$;

$\therefore \angle AEB = \angle BAC$;

and $\therefore CB : AB = AB : BE$.

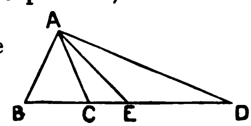


FIG. 424.

Ex. 3. Let DEF be an equilateral △ described about the ⊙ ABC. Through K the centre draw DGB, which bisects EF at rt. ∠s. Draw the tangent $LGM \parallel$ to EF; then LM is the side of a regular hexagon described about the ⊙. Draw the radius KA meeting DE in A.

Now △s DAK, DGL are similar;

$\therefore DA : AK = DG : GL$;

$\therefore 2DA : 2AK = 2DG : 2GL$.

$\therefore$ observing that $DG = KG$, which can easily be shown by joining KL and KM, and proving that $DLKM$ is a rhombus, whose diagonals bisect each other,

$$DE : GB = GB : LM.$$

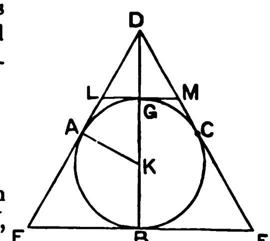


FIG. 425.

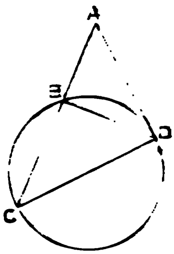


Fig. 426.

Ex. 4. Draw AD a tangent to the $\odot$, and join BD, DC.

Then $\because \angle ADB = \angle ACD$, in alternate segment,

$\therefore \triangle$s ADB, ACD are similar;

$\therefore AB : AD = AD : AC$.

Page 266.

Exercise 1. Since $CA : AD = EA : AB$,

(1) $CA : EA = AD : AB$;	(V. 15.)
(2) $AD : CA = AB : EA$;	(V. 12.)
(3) $AD : AB = CA : EA$.	(V. 15.)

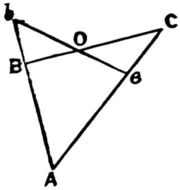


Fig. 427.

Ex. 2. Let O be the intersection of BC, bc.

Then $\because \triangle ABC = \triangle Abc$,

$\therefore \triangle bOB = \triangle cOC$;

$\therefore BO : OC = Oc : Ob$.

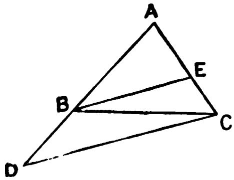


Fig. 428.

Ex. 3. Take AE a mean proportional between AC, CE.

Join EB, and draw $CD \parallel$ to EB.

Then $BE : DC = AE : AC$,

and $BD : AB = EC : AE$.

But, by hypothesis, $AE : AC = EC : AE$;

$\therefore BE : DC = BD : AB$;

and $\angle ABE = \angle BDC$;

$\therefore \triangle ABE = \triangle BCD$.

Ex. 4. Since $AB : AQ$ = duplicate ratio of $AB : AN$;
and $AC : AP$ = duplicate ratio of $AC : AM$,
or, $AP : AC$ = duplicate ratio of $AM : AC$;
and since $AB : AQ = AP : AC$,

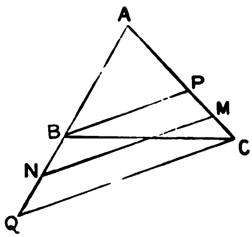


FIG. 429.

$\because$ duplicate ratio of $AB : AN$ = duplicate ratio of $AM : AC$;
$\therefore AB : AN = AM : AC$;
and $\angle$ at A is common to the $\triangle$s ANM, ABC;
$\therefore \triangle ANM = \triangle ABC$.

Page 268.

EXERCISE 1. Draw $CP \perp$ to AB, and $EOF \perp$ to AB.
Then $\triangle$s PCD, CEF are similar;
$\therefore CP : DC = CE : EF$;
$\therefore$ rect. CP, EF = rect. DC, CE;
$\therefore$ rect. CP, AB = rect. DC, CE. (1.)
Again, $EF : CE = DE : OE$;
$\therefore$ rect. EF, OE = rect. CE, DE;
$\therefore$ rect. OE, AB = rect. CE, DE. (2.)
Adding (1) and (2)
rect. OE, AB + rect. CP, AB = sq. on CE;
$\therefore$ 2 area of $\triangle AEB$ + 2 area of $\triangle ACB$ = sq. on CE;
$\therefore$ 2 area of *quadrilateral* $AEBC$ = sq. on CE.

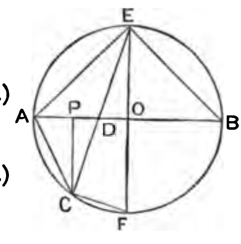


FIG. 430.

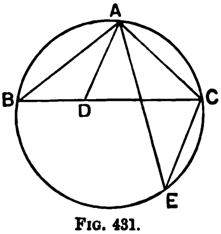


FIG. 431.

Ex. 2. Let ABC be the $\triangle$, AD and AE the lines drawn to meet BC and the $\odot$ described about ABC.

Join EC.

Then $\angle ACE = \angle ADB$, by hypothesis,

and $\angle ABD = \angle AEC$, in same segment,

$\therefore$ $\triangle$s ABD, ACE are similar;

$\therefore AB : AD = AE : AC$;

$\therefore$ rect. $AB, AC =$ rect. AD, AE.

Page 274.

Miscellaneous Exercises.

EXERCISE 1. Let ABC, DEF be the given $\triangle$s.

Produce EF to Q, making $EQ = BC$.

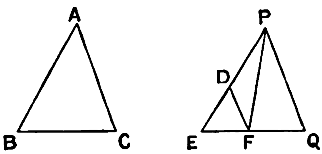


FIG. 432.

Draw $QP \parallel$ to FD, meeting ED produced in P.

Join PF.

Then it may be shown, as in Proposition XIX., that $\triangle EPQ : \triangle EDF$ = duplicate ratio of $EQ : EF$;

and, since $\triangle ABC = \triangle EPQ$,

$\therefore \triangle ABC : \triangle DEF =$ duplicate ratio of $BC : EF$.

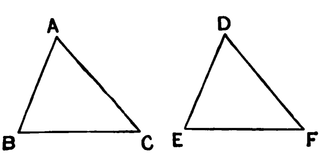


FIG. 433.

Ex. 2. $\triangle ABC : \triangle DEF =$ duplicate ratio of $AC : DF$.

Now $AC : CB = DF : FE$;

$\therefore AC : DF = DF : FE$;

$\therefore AC : EF =$ duplicate ratio of $AC : DF$;

$\therefore \triangle ABC : \triangle DEF = AC : EF$.

Ex. 3. Let ABC be the given $\triangle$.
Construct a square EM = sq. on AB.
Cut off the rectangle $EF = \frac{1}{3}$ of this square.
Describe a square G = rectangle EF.

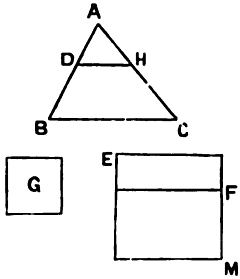


FIG. 434.

Take in AB a pt. D, so that AD = a side of the square last described.
Draw $DH \parallel$ to BC.
Then $\triangle ADH : \triangle ABC$ = sq. on AD : sq. on AB;
$= 1 : 3$.

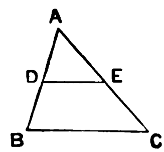


FIG. 435.

Ex. 4. Since $DE = \frac{1}{2} BC$,
$\therefore \triangle ADE = \frac{1}{4} \triangle ABC$;
$\therefore$ quadrilateral $DBCE = 3 \triangle ADE$.

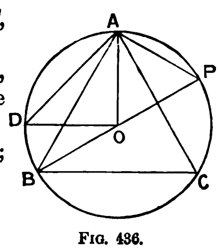


FIG. 436.

Ex. 5. Let O be the centre of the $\odot ABC$, and let ABC be an equilateral $\triangle$. Draw $OD \perp$ to AO; draw the diameter BOP, and join AD, AP; these are the sides of the square and the hexagon.

Then sq. on AD = 2 sq. on AO = 2 sq. on AP;
and sq. on BP = sq. on AB + sq. on AP,
or 4 sq. on AP = sq. on AB + sq. on AP;
$\therefore$ sq. on AB = 3 sq. on AP.
$\therefore$ sq. on AP : sq. on AD : sq. on $AB = 1 : 2 : 3$.

Ex. 6. $\triangle AGF : \triangle AHE = FG : HE,$

$= \text{rect. } FG, HE : \text{sq. on } HE;$

and $\triangle AHE : \triangle ADB = \text{sq. on } HE : \text{sq. on } BD;$

$\therefore \triangle AGF : \triangle ADB = \text{rect. } FG, HE : \text{sq. on } BD;$ (V. 21.)

$\therefore \triangle AGF : \triangle BDC = \text{rect. } FG, HE : \text{sq. on } BD.$ (1.)

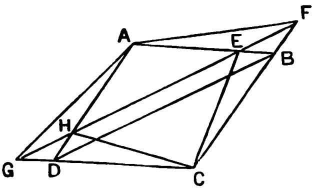


FIG. 437.

Again, $\triangle CFG : \triangle HEC = FG : HE,$

$= \text{sq. on } FG : \text{rect. } FG, HE;$

and $\triangle BDC : \triangle CFG = \text{sq. on } BD : \text{sq. on } FG;$

$\therefore \triangle BDC : \triangle HEC = \text{sq. on } BD : \text{rect. } FG, HE;$ (2.)

$\therefore$ from (1.) and (2.) $\triangle HEC = \triangle AGF.$

Ex. 7. Let ABC, DEF be two $\triangle$s having $\angle ABC = \angle DEF$.

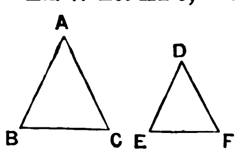


FIG. 438.

Then $\because$

$\triangle ABC : \triangle DEF =$ duplicate ratio of $AB : DE$,

and

$\triangle ABC : \triangle DEF =$ duplicate ratio of $BC : EF$,

$\therefore AB : DE = BC : EF;$

$\therefore \triangle ABC$ is similar to $\triangle DEF$. (VI. 6.)

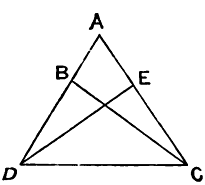


FIG. 439.

Ex. 8. Let the $\triangle$s ABC, ADE have the angle at A common.

Join DC.

Then $\triangle ABC : \triangle ADC = AB : AD,$

$= AB \cdot AC : AD \cdot AC;$

and $\triangle ADC : \triangle ADE = AC : AE,$

$= AD \cdot AC : AD \cdot AE;$

$\therefore \triangle ABC : \triangle ADE = AB \cdot AC : AD \cdot AE.$

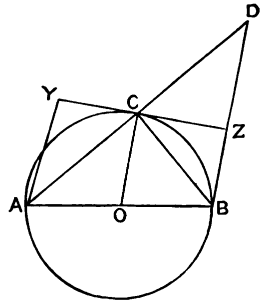


Fig. 440.

Ex. 9. Produce AC to meet BZ produced in D.

Let O be the centre of the ⊙.

Then ∵ AY, OC, BZ are three parallel lines,

∴ $CY=CZ$, since $OA=OB$; (VI. 2, Ex. 2.)

and, since CO is || to BD, and $AO=OB$,

$$\therefore AC=DC.$$

Hence $\triangle ACY = \triangle DCZ$, (I. 4.)

and $\triangle ABC = \triangle DBC$; (I. 4.)

$$\therefore \triangle ABC = \triangle BCZ + \triangle DCZ,$$
$$= \triangle BCZ + \triangle ACY.$$

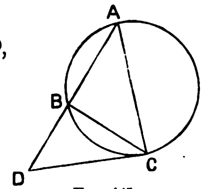


Fig. 441.

Ex. 10. Since $\angle BCD = \angle CAD$, the $\triangle$s BCD, CAD are similar;

$$\therefore AD : CD = CD : DB;$$

$\therefore AD : DB =$ duplicate ratio of $CD : DB$,

$=$ duplicate ratio of $AC : CB$.

Ex. 11. Let $m : n$ be the given ratio.

Let ABC be the given $\triangle$.

Take any point E in AC. Join BE, and make $BE : ED = n : m$.

Join AD, DC. Then DBC is the $\triangle$ required.

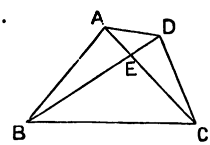


Fig. 442.

For $\triangle AEB : \triangle BEC = AE : EC$,

and $\triangle AED : \triangle CED = AE : EC$;

$\therefore \triangle AEB : \triangle BEC = \triangle AED : \triangle CED$;

$\therefore \triangle ABC : \triangle BEC = \triangle ADC : \triangle CED$;

$\therefore \triangle ABC : \triangle ADC = \triangle BEC : \triangle CED$,

$= BE : ED$,

$= n : m$.

Page 285.

Exercise 1. Area of $\triangle ABC = \frac{1}{2}$ rect. AD, BC.

Now BC can never be greater than the diameter EA;

∴ rect. EA, AD cannot be less than rect. AD, BC;

∴ rect. BA, AC cannot be less than rect. AD, BC;

∴ rect. BA, AC cannot be less than twice area of $\triangle ABC$.

Ex. 2. The line joining OO' bisects AP the common chord of the ⊙s at rt. ∠s. (See. p. 24, Ex. 8.)

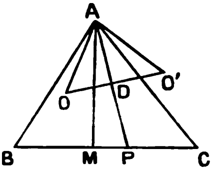


FIG. 443.

Also ∠ AOD, being half the ∠ AOP, is equal to ∠ ABP;

∴ △s AOD, ABM are similar;

and ∴ rect. OD, AM=rect. AD, BM.

Similarly, rect. $O'D$, AM=rect. AD, MC.

∴ rect. OO', AM=rect. AD, BC,

$=\frac{1}{2}$ rect. AP, BC.

Ex. 3. This is precisely the same as Prop. B.

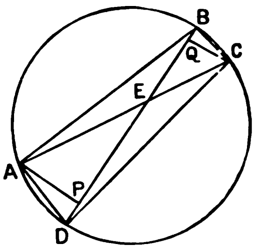


FIG. 444.

Page 286.

EXERCISE. Let $ABCD$ be a quadrilateral inscribed in a ⊙, and let the diagonals intersect at E, so that

∠ AED = ∠ BEC=$\frac{1}{3}$ of a rt. ∠.

Draw AP, CQ ⊥ s to BD.

Then $AE=2AP$ and $EC=2CQ$.

(P. 116, Ex. 3.)

Then area △ $ABD=\frac{1}{2}$ rect. AP, $BD=\frac{1}{4}$ rect. AE, BD,

and area △ $BCD=\frac{1}{2}$ rect. CQ, $BD=\frac{1}{4}$ rect. EC, DB;

∴ area of $ABCD=\frac{1}{4}$ rect. AC, BD;

∴ 4 area of $ABCD$=rect. AB, CD+rect. AD, BC.

Page 288.

EXERCISE. Taking the diagram of the Proposition

$FC:GK=CK:KD$,

$=CK:GF$,

$=KF:AG$ (by similar △s FKC, AGF),

$=DG:AG$,

$=GK:AF$;

and similarly for the other complement.

Page 291.

EXERCISE 1. Let $ABCDE$ be a regular pentagon.
Join AC, BD meeting in P.
Then since AC is $\parallel$ to ED (IV. 11, Ex.),
and BD is $\parallel$ to AE,

$\therefore APDE$ is a rhombus.

Now $\triangle ABC$ is similar to $\triangle BPC$;

$\therefore AC : CB = BC : CP$;

and $CB = AE = AP$;

$\therefore AC : AP = AP : CP$.

Similarly, $BD : PD = PD : BP$.

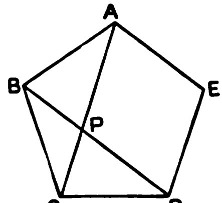


FIG. 445.

Ex. 2. Take the diagram of IV. 10.
Then $BD = AC$, and we have to show that BD is the side of a regular decagon inscribed in $\odot$ BDE.
Now this will be the case if $\angle BAD = \frac{1}{10}$ of four rt. $\angle$s;
and since $\angle BAD = \frac{1}{5}$ of two rt. $\angle$ s,

$\angle BAD$ is $= \frac{1}{10}$ of four rt. $\angle$s.

Miscellaneous Exercises on Book VI.

Page 293.

1. Let O, O' be the centres of the circles, APQ one of the common tangents.

Then $AO : AO' = OP : O'Q$,

$= 1 : 3$;

$\therefore OO' = 2OA$;

$\therefore OA =$ diameter of smaller circle;

and $AO' = \frac{3}{2}\ OO' =$ diameter of larger circle.

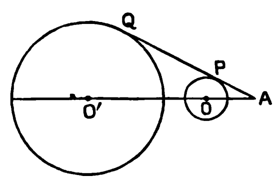


FIG. 446.

2. Draw FE, ED, DF through A, B, C, the angular poin $\triangle ABC$, and $\parallel$ to BC, CA, AB.

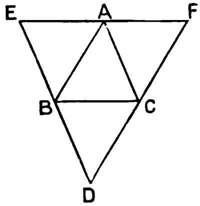


FIG. 447.

Then $\because ABCD$ is a $\square$,

$\therefore \angle BDC = \angle BAC.$

Similarly, $\angle AEB = \angle ACB$,

and $\angle AFC = \angle ABC$.

Hence $\triangle DEF$ is similar to $\triangle ABC$,

and $\because AF = BC$, and $AE = BC$,

$\therefore AE = AF$;

and similarly, $EB = BD$, and $FC = CI$

So also, if lines be drawn through D, E to EF, FD, DE, a $\triangle$ will be formed simil EDF, and having its sides bisected in D, .

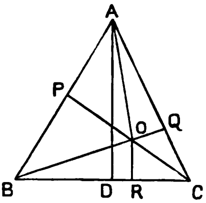


FIG. 448.

3. Let ABC be an equilateral $\triangle$, O any pt. Draw AD, OP, OQ, OR $\perp$s to the opposite s

Then $\triangle ABC : \triangle AOB = AD : OP$;

$\triangle ABC : \triangle AOC = AD : OQ$;

$\triangle ABC : \triangle BOC = AD : OR$;

$\therefore \triangle ABC : \triangle AOB + \triangle AOC + \triangle BOC$

$= AD : OP + OQ +$

$\therefore AD = OP + OQ + OR.$

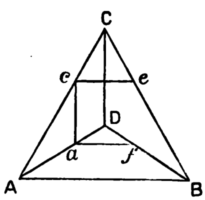


FIG. 449.

4. Let ca be the line $\parallel$ to CD, and let be drawn $\parallel$ to AB.

Then $af : AB = Da : AD$,

$= Cc : CA$,

$= ce : AB$;

$\therefore af = ce.$

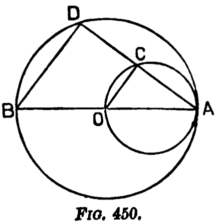


FIG. 450.

5. Let DCA be any chord drawn throug

Then $\angle$s at C and D are right $\angle$s;

$\therefore \triangle$s ACO, ADB are similar;

$\therefore AC : CD = AO : OB$;

$\therefore AC = CD.$

6. Let BC be the given base. Describe on BC the segment of a $\odot$ BAC capable of containing the given vertical $\angle$. Bisect the remaining segment in E. Take D in BC such that $BD=2DC$. Join ED and produce it to A.

Then $\angle BAD = \angle CAD$, and

$\therefore BA : AC = BD : DC = 2 : 1.$

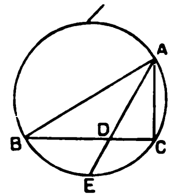


FIG. 451.

7. Since EF is $\parallel$ to AC,

$\therefore AE : EB = CF : FB$;

and $\because EG$ is $\parallel$ to AD,

$\therefore AE : EB = DG : GB$;

$\therefore CF : FB = DG : GB$;

and $\therefore FG$ is $\parallel$ to CD.

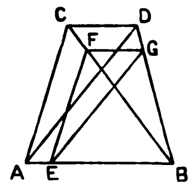


FIG. 452.

8. Let O, P be the pts. in which AD, CF cut BC, AB.

Join AF, DC.

Then $\because \angle$s at P and O are right $\angle$s,

$\therefore \angle LCO = \angle PAL$,

$= \angle OCD$, in same segment;

$\therefore \triangle$s LOC, DOC are equal in all respects;

$\therefore LO = OD$;

and similarly for LE, LF.

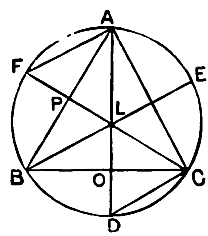


FIG. 453.

9. Since $DE = \frac{1}{3}$ of BC,

$\therefore \triangle ADE = \frac{1}{9}$ of $\triangle ABC$;

and $\triangle DOE = \frac{1}{9}$ of $\triangle BOC$.

Also $\triangle DOB = \triangle EOC$.

Now $\triangle DOB + \triangle DOE + \triangle EOC + \triangle BOC =$ fig. $DBCE$;

$\therefore 2\triangle DOB + 10\triangle DOE = \frac{8}{9}\triangle ABC$;

$\therefore \triangle DOB + 5\triangle DOE = \frac{4}{9}\triangle ABC$. (1.)

Again, $\triangle DOB + \triangle DOE = \triangle ABE - \triangle ADE$,

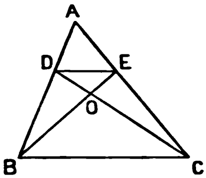


FIG. 454.

$= \frac{1}{3} \triangle ABC - \frac{1}{9} \triangle ABC$,

$= \frac{2}{9} \triangle ABC$.

$\therefore 2 \triangle DOB + 2 \triangle DOE = \frac{4}{9} \triangle ABC$. (

Hence from (1.) and (2.),

$\triangle DOB + 5 \triangle DOE = 2 \triangle DOB + 2 \triangle DOE$

$\therefore 3 \triangle DOE = \triangle DOB$;

$\therefore BO = 3OE$.

Similarly, $CO = 3OD$.

10. Since $\triangle FHG = \triangle AFG - \triangle AFC - \triangle CHG$,

and $\triangle BCH = \triangle ABG - \triangle ABC - \triangle CHG$,

$\therefore \triangle FHG - \triangle BCH = \triangle AFG - \triangle ABG - \triangle AFC + \triangle ABC$;

$\therefore \triangle FHG - \triangle BCH : \triangle ADE$

$= \triangle AFG - \triangle ABG - \triangle AFC + \triangle ABC : \triangle AD$

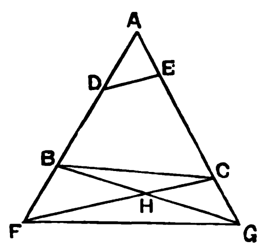


FIG. 455.

$\therefore$, by Ex. 8 on p. 274, $\triangle FHG - \triangle BCH : \triangle ADE$

= rect. AF, AG − rect. AB, AG − rect. AF, AC + rect. AB, AC : rect. AD, A

= rect. AD, AG − rect. AD, AC : rect. AD, AE.

$= AG - AC : AE$,

$= CG : AE$,

$= AE : AE$;

$\therefore \triangle FHG - \triangle BCH = \triangle ADE$;

$\therefore \triangle FHG = \triangle BCH + \triangle ADE$.

11. See p. 251, Ex. 4.

12. Let ABC, DEF be equal $\triangle$s on equal bases, and between the same $\|$s AD, BF.

Draw $MNOP \parallel$ to BF.

Then $\triangle AMN : \triangle ABC$
$=$duplicate ratio of $AN : AC$,
$=$duplicate ratio of $DP : DF$,
(VI. 2, Ex. 1.)
$= \triangle DPO : \triangle DEF$;
$\therefore \triangle AMN = \triangle DPO$.

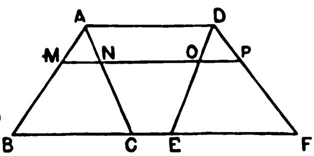


FIG. 456.

13. Bisect AD in T, and draw DER to meet AB in R.
Then since $\angle ACD$ is a rt. $\angle$,
T is the centre of the $\odot$ described about $\triangle ACD$;
$\therefore TC = TA$;
$\therefore TC$ is tangent at C.

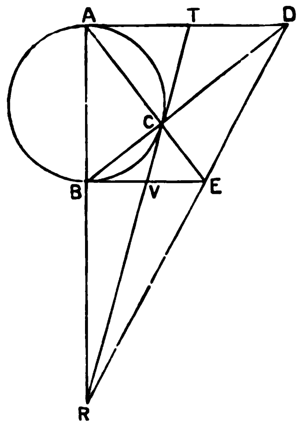


FIG. 457.

Hence CV bisects BE.
$\therefore TV$ produced passes through R.

K

14. Draw any radius OA of the smaller ⊙.
Produce it to B so that $OB=4OA$.
Describe a semicircle on AB, cutting the larger ⊙ in C.

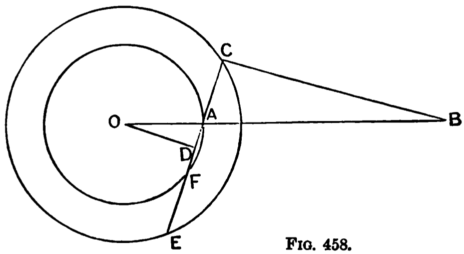


Fig. 458.

Draw $CAFE$ a chord of both circles.
Join CB and draw $OD \perp$ to AF.
Then $\because$ △s AOD, ABC are similar,
$\therefore CD=4DA$; and $\therefore CE=4AF$.

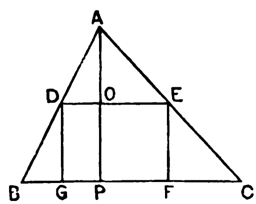


Fig. 459.

15. Draw $AP \perp$ to BC. Bisect BP in G and CP in F.
Then $GF=\frac{1}{2} BC$. Draw GD, FE ⊥s to BC, to meet AB and AC in D, E. Join DE, cutting AB in O.
Then $\because EO=\frac{1}{2} PC$, $\therefore PO=\frac{1}{2} AP$;
$\therefore$ area of rect. $DEFG=$ rect. DG, GF,
$=\frac{1}{4}$ rect. AP, BC,
$=\frac{1}{2}$ area of △.

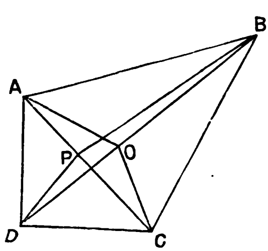


Fig. 460.

16. Let the bisectors of A and C meet in O.
Then $DA : AB=DO : OB$,
$=DC : CB$;
$\therefore DA : DC=AB : CB$.
Now bisect ∠ ADC by DP, meeting AC in P, and join PB.
Then $DA : DC=AP : PC$;
$\therefore AB : CB=AP : PC$;
$\therefore$ PB bisects ∠ ABC.

17. Let $ABCD$ be a quadrilateral described about a ⊙, and let AD be || to BC, and join E, F the pts. of contact of AB, DC.

Then since $AE : EB = DF : FC$;

∴ EF is || to AD and BC.

But tangents make equal ∠s with the chord of contact;

∴ AB, DC are equally inclined to EF;

∴ AB, DC are equally inclined to AD and BC.

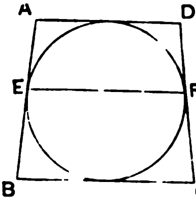


Fig. 461.

18. Let ABC, DBC be △s on the same base BC.

Join AD, and produce it to meet BC produced in E.

Draw AM, DN ⊥s to BE.

Then △s AME, DNE are similar;

∴ $AE : DE = AM : DN$,

= area △ ABC : area △ DBC;

the triangles having the same base, and therefore their areas being proportional to their altitudes.

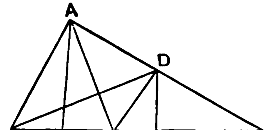


Fig. 462.

19. Let AB, AC be the tangents, and BC the chord of contact.

Draw OD, OE, OF ⊥s from O, any pt. on the Oce, to BC, AC, AB. Join ED, FD, BO, CO.

Then ∵ the ∠s at D, E, F are rt. ∠s, $DOFB$, $DOEC$ can have ⊙s described about them.

∴ ∠ ODE = ∠ OCE, and ∠ OBD = ∠ OFD.

But ∠ OCE = ∠ OBD in alternate segment;

∴ ∠ ODE = ∠ OFD, and similarly it may be shown that ∠ OED = ∠ ODF,

∴ the △s ODE, OFD are similar;

∴ $FO : OD = OD : OE$;

∴ sq. on OD = rect. FO, OE.

(M'Dowell's *Exercises on Euclid*, p. 98.)

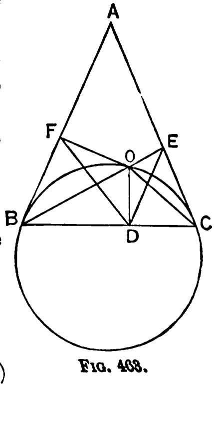


Fig. 463.

20. The two quadrilaterals $EADC$, $CABE$ are similar, for

$\angle EAD = \angle CAB$,
and $\angle ADC = \angle ABE$, and also
$EA : AD = CA : AB$,
and $AD : DC = AB : BE$;
and they have one side EC common,
$\therefore$ they are equal in all respects; (VI. 2
$\therefore AD = AB$, and $AE = AC$,
and $\triangle ABC = \triangle ADE$, and $\triangle ABE = \triangle ACD$.

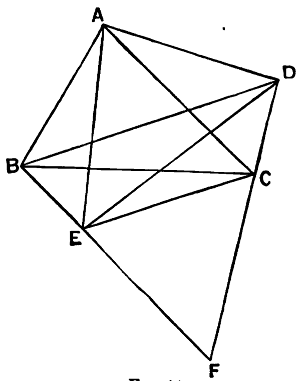


FIG. 464.

Also $BE = CD$, and $\angle ABE = \angle ADC$, and $\angle ABD = \angle ADB$;
$\therefore \angle DBF = \angle BDF$, and $\therefore BF = DF$,
$\therefore EF = FC$, and $\therefore EC$ is $\parallel$ to BD, and the $\triangle$s CFE, BFD a similar.

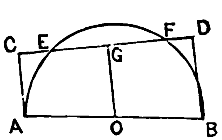


FIG. 465.

21. Let AB be the diameter, and draw A $BD \perp$s to the st. line CD, which meets t $\bigcirc$ce in E and F. Then shall $CE = FD$.

From O the centre draw $OG \perp$ to CF.

Then $CG : GD = AO : OB$; (VI. 2, Ex.

$\therefore CG = GD$;

and $GE = GF$, because OG bisects EF;

$\therefore CE = FD$.

22. Let AB be the given chord.

Take O the centre of the ⊙, and place in the ⊙ a line CD having the given ratio to AB. Draw $OP \perp$ to CD, and draw OQ cutting AB at rt. ∠s and equal to OP. Through Q draw the chord $EF \parallel$ to AB. Then $EF = CD$, being equally distant from the centre, and therefore EF is in the required ratio to AB, and it is $\parallel$ to AB.

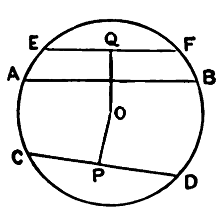


FIG. 466.

23. Since ∠ EAC = ∠ $ACB = \frac{2}{3}$ of a rt. ∠, and ∠ ACE is a rt. ∠,

∴ ∠ $AEC = \frac{1}{3}$ of a rt. ∠;

and ∴ $AE = 2AC$; (See p. 116, Ex. 3.)

and $OF = 2 . OC$,

and $KO = AO = OC$.

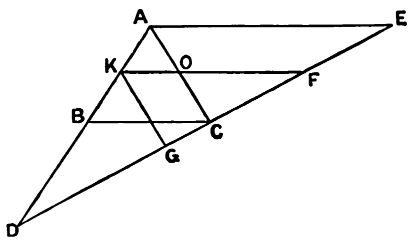


FIG. 467.

Now sq. on KF = sq. on KO + sq. on OF + 2 rect. KO, OF,

= sq. on CO + sq. on OF + 2 rect. CO, OF,

= $\frac{1}{3}$ sq. on CF + $\frac{4}{3}$ sq. on CF + 4 sq. on CO,

= $\frac{1}{3}$ sq. on CF + $\frac{4}{3}$ sq. on CF + $\frac{4}{3}$ sq. on CF,

= 3 sq. on CF,

= 3 sq. on EF;

∴ sum of sqq. on KG, FG = 3 sq. on EF.

24. Let ABC be the △, having the rt. ∠ at B. Draw $CE \perp$ to AC, meeting AB produced in E.

Make ∠ ECD = ∠ CAB, and let CD meet AE produced in D.

Draw $DF \parallel$ to EC, meeting AC produced in F.

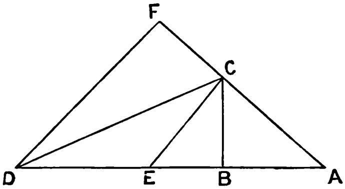


FIG. 468.

Then

$\angle FDC = \angle DCE = \angle CAB.$

$\therefore$ $\triangle$s DCF, ACB are similar,

and $\triangle$s DAF, ACB are similar;

$\therefore FD : DA = BC : CA,$

and $DC : FD = CA : AB$;

$\therefore CD : DA = BC : AB.$

25. Take CD in AD equal to DB, and take BQ in AB produced a third proportional to AC, DB.

With centre Q and distance QD describe a semicircle DRP.

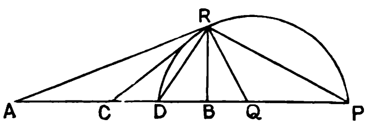


FIG. 469.

Take any pt. R in the Oce.

Then $AC : DB = DB : BQ$;

$\therefore AD : DB = DQ : BQ$; (V.16.)

$\therefore AD : DQ = DB : BQ$;

$\because AQ : DQ = DQ : BQ$; (V.16.)

$\therefore AQ : QR = QR : BQ.$

Hence ARQ, RQB are similar $\triangle$s; (VI. 6.

$\therefore AR : RB = AQ : RQ$;

$\therefore AR : RB = AQ : DQ,$

$= DQ : BQ,$

$= AD : DB$;

$\therefore DR$ bisects $\angle ARB$;

and $\therefore$, since RP is $\perp$ to DR,

$AP : PB = AD : DB.$

26. Let G be the point of contact of the circles ABG, DEG, of which AB, DE are parallel diameters, and O, C the centres.

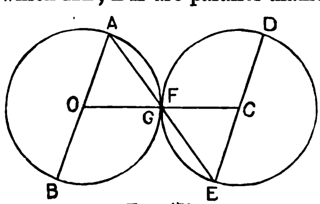


FIG. 470.

Then OC passes through G.

Let OC meet AE in F.

Then $\triangle$s OAF, CEF are similar;

$\therefore AO : CE = OF : FC$;

$\therefore AO + CE : CE = OF + FC : FC$;

$\therefore OC : CE = OC : FC$;

$\therefore FC = CE = CG$;

$\therefore F$ and G are coincident.

27. Let ⊙s ABE, CDE touch each other and also touch the st. line AC.

Draw the diameters AB, CD, and join AD, BC; these lines (as is proved in Ex. 26) pass through E, the point of contact of the ⊙s.

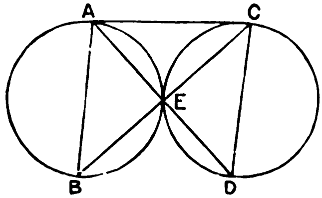


FIG. 471.

Then $\angle\, CAD = \angle\, ABE$, in alternate segment;

$\therefore$ △s ABC, CAD are equiangular;

$\therefore BA : AC = AC : CD.$

28. Let AEB, AFC be the circles; O, P their centres.

Draw the common diameter $AOBPC$.

Draw AEF any chord of both circles, and join BE, CF.

Then AEB and AFC are rt. angles;

$\therefore EB$ is $\parallel$ to FC;

$\therefore FA : AE = CA : AB,$

$= 3 : 1.$

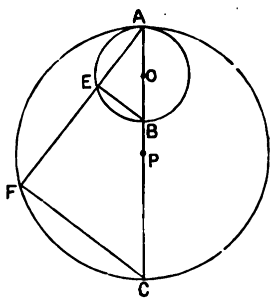


FIG. 472.

29. Make $\angle\, BCP = \angle\, BCD$. Then P is the point required.

For, since CB bisects $\angle\, DCP$,

$PB : BD = CP : CD$;

and, since AC bisects the external $\angle$ of △PCD,

$\therefore PA : AD = CP : CD.$

Hence $PA : AD = PB : BD$;

and $\therefore PA : PB = AD : DB.$

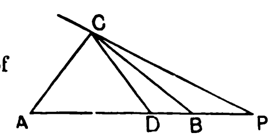


FIG. 473.

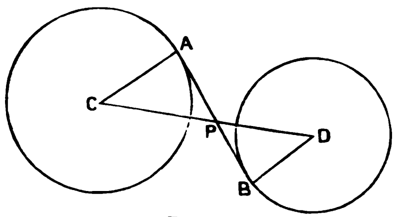


FIG. 474.

30. Let C and D be the centres of the ⊙s, AB a common tangent at A and B.

Let CD and AB intersect in P.

Then, since ∠s at A and B are rt. ∠s, the △s APC, BPD are similar;

$\therefore CP : PD = AC : BD$
$= 2AC : 2BD.$

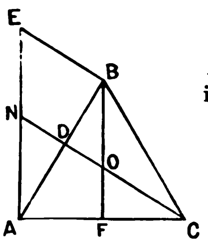


FIG. 475.

31. Let ABC be an equilateral △.

Draw $BE \perp$ to AB, $AE \perp$ to AC, and BF, CD ⊥s to AC, AB, and let CD produced meet AE in N.

Then $ENOB$ is a ▱, and $\therefore NO = EB$.

Now since OF is ∥ to AN,

$NO : OC = AF : FC$;

and ∴, since $AF = FC$,

$NO = OC$; and $\therefore EB = CO =$ radius of circumscribing ⊙.

32. The parallelograms being equal to each other, the rectangles on the same bases and with the same altitudes are equal.

∴ the altitudes are inversely proportional to the bases, that is, to the diagonals of the parallelogram.

33. Since $\triangle FAG : \triangle FGD = AG : GD$,

and $\triangle EAG : \triangle EGD = AG : GD$;

$\therefore \triangle EAG : \triangle FAG = \triangle EGD : \triangle FGD$;

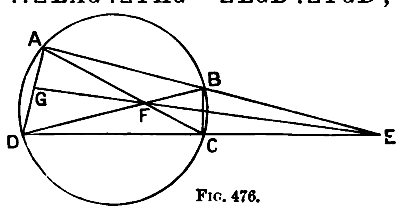


FIG. 476.

$\therefore \triangle EAF : \triangle FAG = \triangle EFD : \triangle FGD$; (V. 25.)

$\therefore \triangle EAF : \triangle EFD = \triangle FAG : \triangle FGD$;

$\therefore \triangle EAF : \triangle EFD = AG : GD$.

Hence $AG : GD$ = rect. EA, FA : rect. ED, DF. (See p. 274, Ex. 8.)
But from similar △s AFB, DFC,

$$AB : CD = AF : FD;$$

∴ rect. EA, AB : rect. ED, DC = rect. EA, AF : rect. ED, FD,
$= AG : GD$.

34. Let AOB be the diameter of a ⊙, and O the centre.
Draw AD, BE tangents at A and B, meeting a tangent at any pt. C in D and E.
Then △s COE, BEO are equal.
Since ∠ ECO = ∠ OBE = a rt. angle, and $OB = OC$;

∴ ∠ COE = ∠ EOB.

So also ∠ COD = ∠ DOA.
Hence ∠ DOE is a rt. angle;
and ∵ in the right-angled △ DOE, OC is drawn ⊥ to DE,

∴ $DC : CO = CO : CE$.

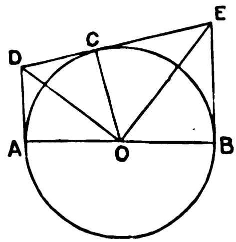


FIG. 477.

35. Let ABC be the △, and draw AD, CE, making ∠ DAC = ∠ ABC,
and ∠ ECA = ∠ ABC.
Then △s ADC, BAC are similar,
and △s AEC, BAC are similar;
∴ △s ADC, AEC are similar;
∴ $AD : AC = AC : CE$;
∴ rect. AD, CE = sq. on AC.

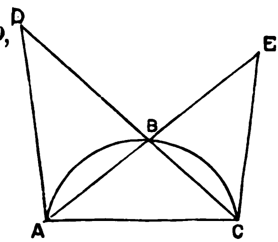


FIG. 478.

36. Let $ABCD$ be the given parallelogram, EF the given altitude.
Draw EH, FG ⊥s to EF.
Make ∠ GFO = ∠ ADC, O being a pt. in EH.

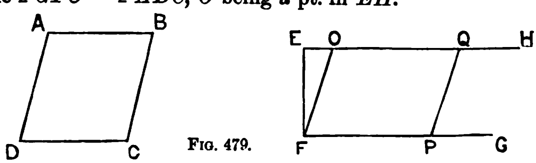


FIG. 479.

Take FP a fourth proportional to OF, AD, DC.
Draw PQ ∥ to OF.
Then ▱ $OFPQ$ = ▱ $ABCD$; (VI. 14.)

∴ $OFPQ$ is the ▱ required.

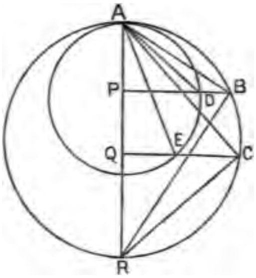


FIG. 480.

37. Let AR be the line drawn from A, the point of contact, through the centres of the ⊙s.

From P and Q draw PDB, QEC ⊥s to AR.

Then $AR : AB = AB : AP$; (VI. 12.)

$\therefore AR : AP =$ duplicate ratio of $AR : AB$.

Similarly, $AQ : AR =$ duplicate ratio of $AC : AR$;

$\therefore AQ : AP =$ duplicate ratio of $AC : AB$;

so also, $AQ : AP =$ duplicate ratio of $AE : AD$;

$\therefore AC : AB = AE : AD$.

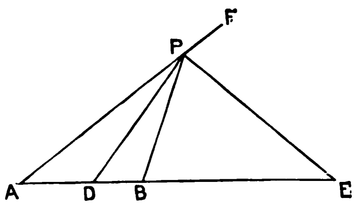


FIG. 481.

38. Let AB be the base, and E the point in which the line bisecting the exterior angle meets the base.

Divide AB in D, so that $AD : DB = AE : BE$; then D is the point in which the line bisecting the interior vertica angle meets the base, and is a fixed point.

Also, if P be the vertex, $\angle DPB = \frac{1}{2} \angle APB$, and $\angle EPB = \frac{1}{2} \angle BPF$;

$\therefore \angle DPE = \frac{1}{2} \angle APB + \frac{1}{2} \angle BPF =$ a rt. angle;

$\therefore$ locus of P is the circle on DE as diameter.

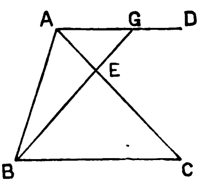


FIG. 482.

39. Let ABC be the given △.

Draw AD ∥ to BC. Divide AC in E so that $AC : CE$ in the given ratio.

Join BE and produce it to meet AD in G.

Then △s AEG, BEC are similar;

$\therefore EG : BE = EA : CE$;

$\therefore BG : BE = AC : CE$; (V. 16.)

$\therefore BG : BE =$ the given ratio.

40. Since $\triangle$s ABD, ACD are similar, and $\angle$ at D is common to both,

$\therefore \angle ACD = \angle BAD$;
and $\angle DAC = \angle ABC$;
Hence $AD : CD = BD : AD$;
$\therefore$ sq. on AD = rect. CD, BD
$\therefore$ DA touches the $\odot$ described about $\triangle ABC$.

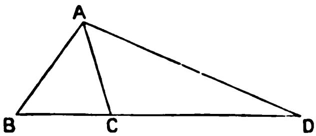


Fig. 483.

41. Let T be any pt. in the line through $P \perp$ to AB,
PR, PQ the equal tangents from P,
TS, TV the tangents from T.

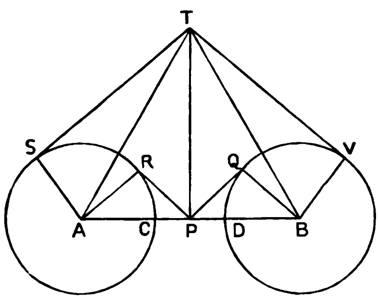


Fig. 484.

Then sqq. on TS, AS, BD = sqq. on TA, BD,
= sqq. on TP, AP, BD,
= sqq. on TP, AR, RP, BD,
= sqq. on TP, AC, PQ, BQ,
= sqq. on TP, AC, BP,
= sqq. on BT, AC,
= sqq. on BV, TV, AC;
$\therefore$ sq. on TS = sq. on TV.

42. Let ABC be the triangle, OP, OQ, OR the three lines.
Describe on AB a segment of ⊙ containing $\angle\, QOP$,
on AC „ „ $\angle\, POR$,
on BC „ „ $\angle\, QOR$.

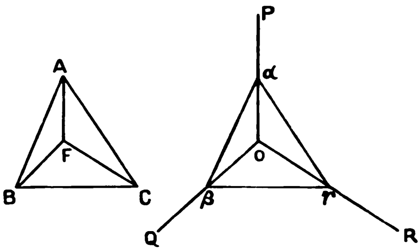


Fig. 485.

These will intersect in one pt. F, since the sum of the three $\angle$s is four right $\angle$s. Then take $O\alpha$, $O\beta$, $O\gamma$ equal to AF, BF, CF respectively, and $\triangle\ \alpha\beta\gamma$ is not only similar but equal to $\triangle\ ABC$, and any number of similar $\triangle$s may be formed by starting at any point in OP and drawing straight lines $\parallel$ to $\alpha\gamma$, $\gamma\beta$, $\beta\alpha$.

43. Let BE, DF, the $\perp$s on AD, AB, or these produced, meet in O. Join OA, and produce it to meet BD in G.

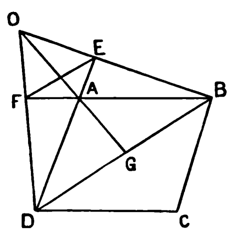


Fig. 486.

Then shall OG be $\perp$ to BD.

For, if FE be joined, since a ⊙ can be described about $OFAE$, the angles at E and F being right $\angle$s,

$\angle\, FOG = \angle\, FED$ in the same segment.

And since a ⊙ can be described about $DFEB$,

$\angle\, FED = \angle\, FBD$ in the same segment;

$\therefore\ \angle\, FOG = \angle\, FBD$;

and $\angle\, OAF = \angle\, BAG$;

$\therefore\ \angle\, AGB = \angle\, OFA =$ a right $\angle$.

44. Let $ABCD$ be a quadrilateral inscribed in a ⊙; join AC, BD.
Make $\angle\, ABF = \angle\, DBC$, then $\angle\, ABD = \angle\, EBC$, and $\therefore\ AD = FC$.
Then since $\angle\, ADB = \angle\, ECB$ in same segment, the $\triangle$s ABD, EBC are similar;

$\therefore$ rect. BD, $BE =$ rect. AB, BC.

Again, $\angle ECF = \angle ABF = \angle DBC$; and $\angle EFC = \angle CDB$;

$\therefore \triangle$s ECF, BDC are similar;

and $\therefore$ rect. $FE, BD =$ rect. $CF, CD =$ rect. AD, CD.

Hence rect. $BF, BD =$ rect. $AB, BC +$ rect. AD, CD.

Similarly, by taking $\angle BCG = \angle ACD$, we may show that rect. $CG, CA =$ rect. $AB, AD +$ rect. CB, CD.

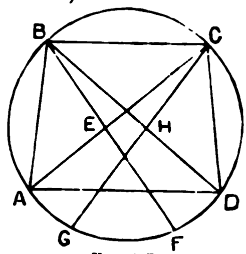


FIG. 487.

And since $\angle BCG = \angle ACD$,

$\therefore$ arc $BG =$ arc $AD =$ arc FC;

$\therefore$ arc $BAF =$ arc GFC;

$\therefore BF = GC$;

$\therefore$ rect. $CG, CA =$ rect. BF, CA.

Then $AC : BD =$ rect. AC, BF : rect. BD, BF,

$=$ rect. $AB, AD +$ rect. CB, CD : rect. $AB, BC +$ rect. AD, CD.

45. Let ABC be the $\triangle$, and $BDEC$ the square.

Let AD, AE cut the base in P, Q.

Draw $AF \perp$ to BC. Then $PF = QF$.

Now, by similar $\triangle$s APF, DPB,

$PF : PB = AF : BD$,

$= AF : BC$;

$\therefore PF : AF = PB : BC$;

$\therefore 2PF : 2AF = PB : BC$;

$\therefore PQ : 2AF = PB : BC$;

$\therefore PQ : PB = 2AF : BC$.

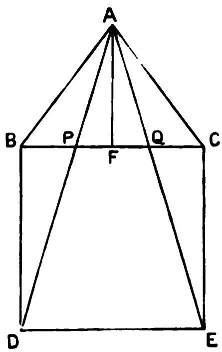


FIG. 488.

46. Since rect. $DA, BE =$ sq. on AC,

$\therefore DA : AC = AC : BE$,

$= CB : BE$;

and $\angle DAC = \angle CBE$;

$\therefore \triangle DAC$ is similar to $\triangle CBE$.

(VI. 6.)

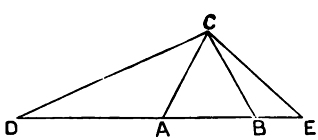


FIG. 489.

47. Bisect the $\angle$s at B and A by st. lines meeting in O, and let BO meet AC in b. Then $\angle\, bBA = \frac{1}{2} \angle\, CBA = \angle\, BAC$; and $Bb = Ab$; and similarly, $Ca = Aa$.

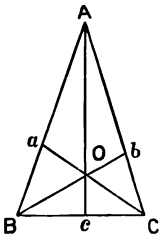


FIG. 490.

Now since AO bisects $\angle\, BAb$,

$$\therefore\ BA : BO = Ab : Ob;$$

$$\text{and } \therefore\ BA - BO : BO = BO : Ob. \qquad (1.)$$

Again, $\because$ CO bisects $\angle\, BCb$,

$$\therefore\ BC : BO = Cb : Ob,$$

$$= AB - Bb : Ob;$$

$$\therefore\ BC + BO : BO = AB - BO : Ob. \qquad (2.)$$

From (1.) and (2.),

$$BC + BO : BA - BO = AB - BO : BO;$$

$$\therefore\ BC + BA : BA - BO = AB \,.\, BO;$$

$$\therefore\ BC + BA : AB = BA - BO : BO;$$

$$\therefore\ BC + 2AB : AB = AB : BO;$$

i.e. perimeter : side = side : distance of cent. of inscr. $\odot$ from B.

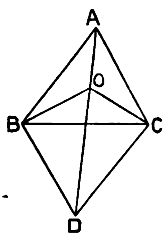


FIG. 491.

48. Let D be the centre of the $\odot$ escribed to ABC, which touches BC externally. Then AOD is a straight line, because AD, AO both bisect the angle BAC; also, OB, OC bisect the interior angles at B and C; and BD, CD bisect the exterior angles at B and C; therefore OBD, OCD are right $\angle$s.

$\therefore$ AO passes through the centre of the $\odot$ described about $BOCD$.

(*Solutions of Senate-House Problems for* 1878.)

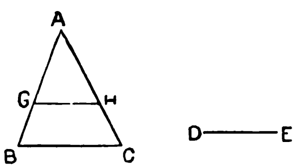


FIG. 492.

49. Let ABC be the $\triangle$. Take DE a third proportional to AB, BC. Divide AB in G so that $AG : GB = AB : DE$, and draw $GH \parallel$ to BC.

Then $AG : GB =$ duplicate ratio of $AB : BC$;

$\therefore$ $AG : GB =$ duplicate ratio of $AG : GH$;

$\therefore$ $AG : GH = GH : GB$.

50. Let ABC be an equilateral △ inscribed in a ⊙.

Take D, E the middle pts. of arcs ADB, AEC.

Let DE cut AB, AC in the pts. P, Q.

Join DC, AE, DB.

Then ∵ ∠ CDE = ∠ DCB, subtended by an equal arc,

∴ DE is || to BC.

Then ∵ ∠ DPB = ∠ APE,

and ∠ DBP = ∠ AEP,

and $DB=AE$, subtended by an equal arc;

∴ $DP=PA$, and similarly $EQ=QA$.

Hence, since APQ is an equilateral △,

$PQ=DP=QE$.

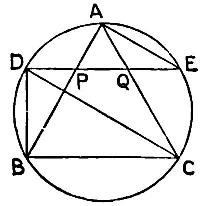


Fig. 493.

51. ∠ $ACD=\frac{4}{3}$ of a right ∠;

∴ ∠ CAD = ∠ $CDA=\frac{1}{3}$ of a right ∠;

∴ ∠ BAD is a right ∠.

Also, ∠ $ACE=\frac{1}{3}$ of a right ∠;

∴ ∠ $AEC=\frac{1}{3}$ of a right ∠;

∴ $AE=AC$; and ∠ DAE is a right ∠;

∴ BAE is a straight line, and $BE=BD$;

∴ BED is an equilateral △, and ∴ $EC=DA$.

Then rect. DA, CE

=sq. on CE,

=sq. on BE − sq. on BC,

=3 sq. on BC,

=2 rect. BC, BC + sq. on BC,

=2 rect. AC, AC + sq. on BC,

=rect. DE, AC + sq. on BC.

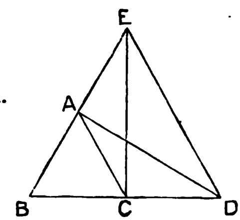


Fig. 494.

52. Since $CE:EA=BE:ED$,

∴ rect. CE, ED=rect. EA, BE;

∴ a ⊙ described round △ BCD will pass through A.

For, if not, let it cut AB in M.

Then rect. CE, ED=rect. BE, EM,

∴ $EM=EA$, which is absurd.

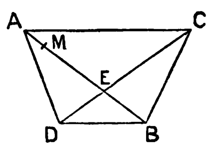


Fig. 495.

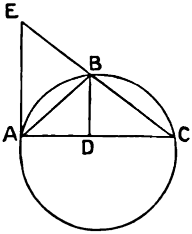


FIG. 496.

53. Let ABC be the $\triangle$, and AE the tangent at A.

Draw $BD \parallel$ to AE.

Now $\angle EAB = \angle BCD$,

and $\angle AEB = \angle DBC$;

$\therefore$ $\triangle$s AEB, DCB are similar;

$\therefore AE : AB = CB : CD$.

Also $AC : AE = DC : DB$ (by similar $\triangle$s ACE, DCB),

$\therefore AC : AB = CB : DB$. (V. 21.)

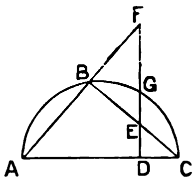


FIG. 497.

54. Let ABC be the $\triangle$, $DF \perp$ to AC.

Then $\angle BFE = \angle ECD$;

$\therefore$ $\triangle$s ADF, EDC are similar;

$\therefore DF : DA = DC : DE$.

But $DA : DG = DG : DC$; (VI. 13.)

$\therefore DF : DG = DG : DE$. (V. 21.)

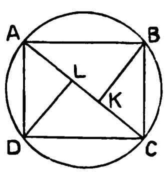


FIG. 498.

55. By VI. C,

rect. BK, diameter = rect. AB, BC;

rect. DL, diameter = rect. AD, DC;

$\therefore BK : DL =$ rect. AB, BC : rect. AD, DC.

56. Let ABC be the $\triangle$, and $DBFE$ the rectangle.

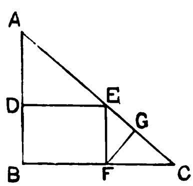


FIG. 499.

Draw $FG \perp$ to AC.

Then by similar $\triangle$s ADE, EGF,

$AD : AE = EG : EF$;

$\therefore$ rect. AD, EF = rect. AE, EG, (1.)

and, by similar $\triangle$s ADE, FGC,

$DE : AE = GC : FC$;

$\therefore$ rect. DE, FC = rect. AE, GC. (2.)

Hence from (1.) and (2.), observing that $BD = EF$, and $BF = DE$,

rect. AD, DB + rect. BF, FC = rect. AE, EC.

57. Let OAB be an isosceles $\triangle$, and let the circle cut the base in C, D, and the sides in F, E.

Then $\because OE : OB = OF : OA$;

$\therefore FE$ is $\parallel$ to AB.

Hence if CE be joined, $\angle FEC = \angle ECD$;

and $\therefore$ arc FC = arc ED;

$\therefore \angle FOC = \angle DOE$;

and $OA = OB$, $OC = OD$, and $\therefore AC = DB$. (I. 4.)

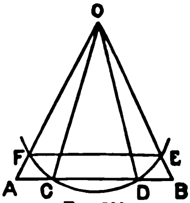


Fig. 500.

58. Let AC be the diameter through A, and let MP be drawn $\perp$ to AC produced.

Then the $\triangle$s ABC, APM are similar;

$\therefore AB : AC = AP : AM$;

$\therefore$ rect. AB, AM = rect. AC, AP;

$\therefore$ rect. AC, AP is constant;

$\therefore P$ is a fixed point;

and the locus of M is a straight line through P $\perp$ to AP.

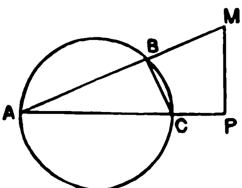


Fig. 501.

59. In the $\triangle ABC$ let AB be greater than AC.

Draw BE, CF $\perp$s on AC, AB.

Then $\because$ $\triangle$s ABE, ACF are similar;

$\therefore AB : BE = AC : CF$;

$\therefore AB : AB - BE = AC : AC - CF$;

$\therefore AB : AC = AB - BE : AC - CF$;

$\therefore AB - BE$ is greater than $AC - CF$;

$\therefore AB + CF$ is greater than $AC + BE$.

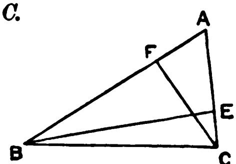


Fig. 502.

60. Let Ap, Aq, the $\perp$s from A on the external bisectors of the $\angle$s at B and C, meet BC produced in P and Q.

The $\triangle$s ABp, PBp are evidently equal,

$\therefore PB = AB$, and similarly, $CQ = AC$.

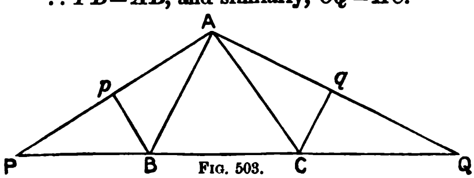


Fig. 503.

$\therefore PQ$ = the perimeter of $\triangle ABC$;

and as p and q are the middle points of AP, AQ;

$\therefore pq$ = *half* the perimeter of $\triangle$ ABC.

61. Let A, B be fixed points on a ⊙, and P a movable pt.; also let p be the centre of ⊥s of △ APB.

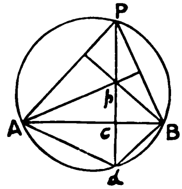


FIG. 504.

Let Pp produced meet AB in c, and the ⊙ in a.
Then ∠ ApB = supplement of ∠ APB,
= ∠ AaB.

∴ if △ AaB be turned round AB, a will coincide with p;

∴ $ca = cp$.

∴ $Pp = cP - ca$ = twice the distance of the centre of the ⊙ from AB.

Hence Pp is constant for different positions of P, and being drawn in a constant direction, the △s PQR, pqr must be equal, similar, and similarly placed, as one will coincide with the other when moved ∥ to itself through a constant distance.

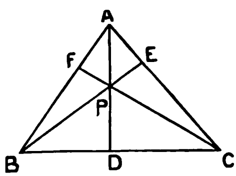


FIG. 505.

62. The △s APF, ABD are similar;
∴ $AP : FP = AB : BD$;
∴ rect. AP, BD = rect. FP, AB.
Similarly, rect. AP, CD = rect. EP, AC.
∴ rect. AP, BC = rect. FP, AB + rect. EP, AC.

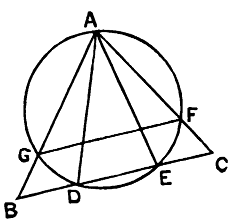


FIG. 506.

63. About △ ADE describe a ⊙, cutting AB, AC, produced if necessary, in G and F. Join FG.

Then ∵ ∠ GAD = ∠ FAE,
∴ arc GD = arc EF;
∴ ∠ GFD = ∠ FDE;
∴ FG is ∥ to BC;
∴ $AB : AC = BG : CF$;
∴ sq. on AB : sq. on AC = rect. AB, BG : rect. AC, CF;
= rect. BD, BE : rect. CD, CE.
(III. 36.)

64. Let A, B, C be the given points, and $l : m : n$ be the ratio of the perpendiculars.

Take a pt. D in AB so that $AD : BD = l : m$, and a pt. E in AC so that $AE : CE = l : n$.

Then DE is the line required, since $\perp$ s on it from A and B are as $AD : BD$, *i.e.* as $l : m$; and those from A and C are as $l : n$; and $\therefore$ those from B and C are as $m : n$.

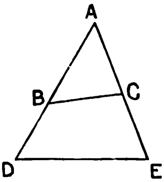


FIG. 507.

65. Let ABC, DEF, GHK be three similar $\triangle$ s.

Take MN a fourth proportional to BC, EF, HK; and on MN describe a $\triangle$ LMN similar to $\triangle$ ABC.

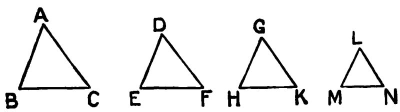


FIG. 508.

Then, by VI. 24,

$$\triangle ABC : \triangle DEF = \triangle GHK : \triangle LMN.$$

66. Let ABC be a $\triangle$, DEF the middle pts. of its sides.

Let the bisectors of the sides meet in O.

Join FE.

Then AOB, EOF are similar $\triangle$ s;

$$\therefore AO : OE = AB : EF,$$
$$= 2 : 1$$

Similarly, $BO : OF = 2 : 1$, and $CO : OD = 2 : 1$.

Draw $AN \perp$ to BC, and $OP \perp$ to AN.

Then $\triangle ABC : \triangle OBC = AN : PN$,

$$= AE : OE,$$
$$= 3 : 1.$$

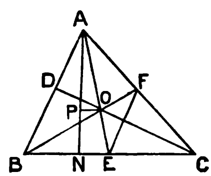


FIG. 509.

67. Let OP, OQ be the given lines, C a pt. such that if CP, CQ be drawn $\perp$ to OP, OQ, the line PQ is constant. Then since PQ is constant, and $\angle POQ$ is fixed, the circle circumscribing OPQ is of constant diameter, and OC is the diameter of this $\odot$. Hence the locus of C *is a* $\odot$ *with centre* O.

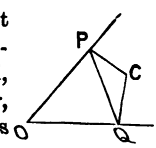
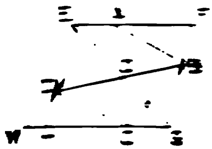


FIG. 510.

68. Let $EFGH$ be one of the ▱s, and let AD produced meet CH produced in M. Let EB, FG intersect in O.

$$\text{Then } \angle BOG = \angle DMH$$



$$= \angle EAD;$$

and $\angle AED = \angle BGC$, and $AD = BC$;

$\therefore ED = BG$, and since △s EOD, BOG are equiangular,

$\therefore OB = OD$ and $OE = OG$;

$\therefore O$ is the point of bisection of diagonals of $ABCD$, $EFGH$.

Fig. [illegible]

Hence the diagonals of all the ▱s pass through O.

69. Join RT.

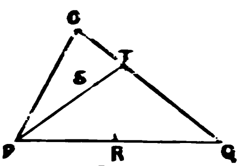


Fig. 512.

$$\text{Then } \triangle ORQ = \triangle OPR;$$

$$\text{and } \triangle QRT = \triangle PTR;$$

$$\therefore \triangle ORQ - \triangle QRT = \triangle OPR - \triangle PTR;$$

$$\therefore \triangle ORT = \triangle OSP - \triangle STR;$$

$$\therefore 2\triangle STR = \triangle OSP - \triangle STR;$$

$$\therefore 3\triangle STR = \triangle OSP;$$

$$\therefore 6\triangle STR = 2\triangle OSP,$$

$$= \triangle OPR,$$

$$= \triangle ORQ;$$

$$\therefore 3\triangle ORT = \triangle ORQ;$$

$$\therefore OQ = 3OT.$$

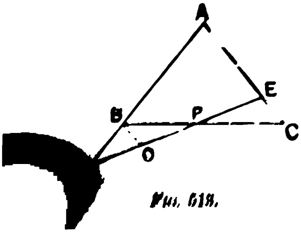


Fig. 513.

70. Let P be the middle pt. of BC.

Draw $BO \parallel$ to AC, and meeting EPD in O

Then $\because BP = PC$, $\therefore OB = EC$;

and, since BO is $\parallel$ to AC,

$$AD : BD = AE : BO,$$

$$= AE : EC.$$

71. Area of $\triangle ABC$

$=\frac{1}{2}$ rect. CP, AB,

$=\frac{1}{2}$ rect. $CP, 2AQ$,

$=$ rect. CP, CQ,

$=$ a constant.

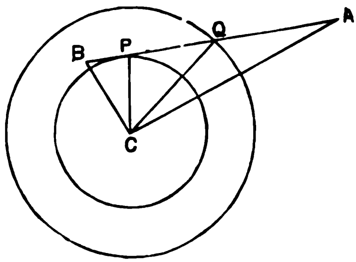


FIG. 514.

72. Let PD meet the concave Oce in R.

Then since CAP, CBP are rt. ∠s, a ⊙ described on PC as diameter passes through A and B. And since PFC is a rt. ∠, F is a pt. in the Oce of this ⊙.

Then rect. $FP, FE =$ rect. $FE, EP +$ sq. on EF, (II. 3.)

$=$ rect. $AE, EB +$ sq. on EF, (III. 35.)

$=$ rect. $RE, ED +$ sq. on EF, (III. 35.)

$=$ sq. on FD. (II. 5.)

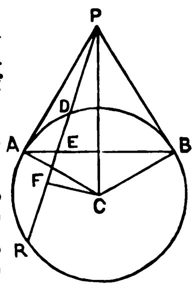


FIG. 515.

73. Let α, β, γ be the points of contact of the inscribed ⊙, and let the three ⊙s be drawn as described.

Then $A\gamma$ and $A\beta$ are equal, being tangents to the inscribed ⊙. Join AG; it must be a tangent to the two ⊙s $G\gamma$, $G\beta$, since the rectangle under the segments of any line drawn to these ⊙s from A must be equal to the squares on $A\gamma$, $A\beta$, which are equal, and therefore AG cannot meet these ⊙s in any other point than G.

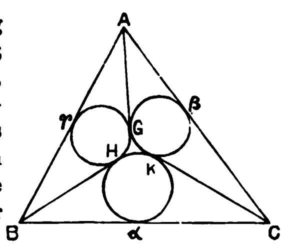


FIG. 516.

Similarly with regard to BH, CK.

But tangents to the three ⊙s at G, H, K meet in a point;

$\therefore AG, BH, CK$ meet in a point.

74. $BH : BK = EH : AK$,
$= FL : AK$,
$= CL : CK$;
$\therefore$ rect. BH, CK = rect. CL, BK.

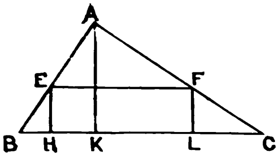


FIG. 517.

Now sq. on AC = rect. BC, CK = rect. BH, CK + rect. CH, CK;
and sq. on AB = rect. BC, BK = rect. BL, BK + rect. CL, BK;
$\therefore$ sq. on AC − sq. on AB = rect. CH, CK − rect. BL, BK.

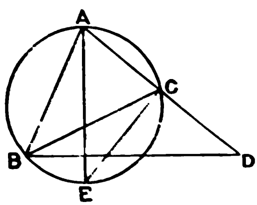


FIG. 518.

75. Let AE be the diameter; join BC, CE.

Then $\angle ABD + \angle BAE$ = a rt. $\angle$;
and $\angle ACB + \angle BCE$ = a rt. $\angle$;
also $\angle BAE = \angle BCE$ in the same segment;
$\therefore \angle ABD = \angle ACB$.
Hence $\triangle$s ABD, ACB are similar;
$\therefore AD : AB = AB : AC$.

76. Let ABC be the fixed chord, and APQ the movable chord, and let PB, QC intersect in R.

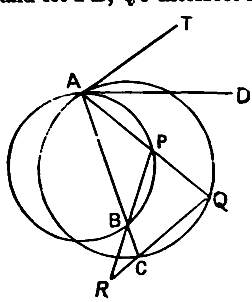


FIG. 519.

Draw TA, DA tangents to the ⊙s at A.
Then $\angle TAQ = \angle ACQ$ in alternate segment;
and $\angle DAP = \angle ABP$ in alternate segment;
$\therefore \angle TAD = \angle ACQ - \angle ABP$,
$= \angle ACQ - \angle RBC = \angle BRC$.
Now $\angle TAD$ is constant, and $\therefore \angle BRC$ is constant.
Hence the locus of R is a circle passing through BC.

77. $AE : ED = BC : CD$. (1.

Also, $AE : DA = BC : BD$,

$= AB : BD$,

$= EC : CD$; (2.)

∴, compounding the ratios in (1.) and (2.),

sq. on AE : rect. ED, DA = rect. BC, CE : sq. on CD.

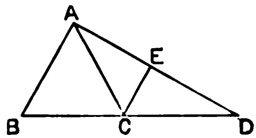


FIG. 520.

78. Bisect BC in E; then E is the centre of the ⊙ described about △ ABC.

Draw $ED \parallel$ to BA, then ED bisects AC.

Then ∠ $AEB = 2$ ∠ ACE. (III. 20.)

Now AE is greater than AD;

∴ AE is greater than AB;

∴ ∠ ABE is greater than ∠ AEB;

∴ ∠ ABE is greater than 2 ∠ ACE.

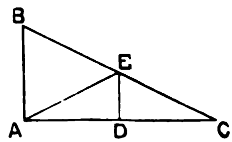


FIG. 521.

79. Describe a ⊙ about the △ ABC, and produce AB to D, making $BD = AB$.

From D draw $DE \parallel$ to BC, meeting the Oce in E.

Join AE, cutting BC in F.

Then ∵ $AB = BD$, ∴ $AF = FE$.

Now rect. BF, FC = rect. AF, FE = sq. on AF;

∴ $BF : AF = AF : FC$.

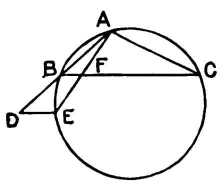


FIG. 522.

80. Since △s FAG, DAE are similar,

∴ $AG : AE = FA : AD$;

and since △s FAC, DAB are similar,

∴ $FA : AD = CA : AB$.

Hence $AG : AE = CA : AB$;

∴ EG is $\parallel$ to BC.

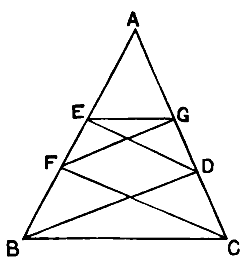


FIG. 523.

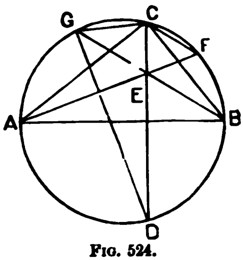


Fig. 524.

81. $\angle AFC = \angle ABC$, in same segment;
$\angle AFD = \angle ACD$, in same segment,
$= \angle ABC$, because CD cuts AB at rt. $\angle$s;
$\therefore \angle AFC = \angle AFD$.
Again, $\angle DGB = \angle DCB = \angle CAB = \angle CGB$;
$\therefore \angle$s at F and G are bisected by AF, BG;
$\therefore CF : FD = CE : ED$, (VI. 3.)
$= CG : GD$.

82. Let p, f, g be the pts. of tangency.

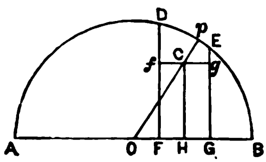


Fig. 525.

Then sq. on CH
$=$ sq. on $OC -$ sq. on OH,
$=(OC+OH).(OC-OH)$, (II. B. p. 84.)
$=(OC+OF+FH).(OC-OG+HG)$.
Now $FH=HG=fC=gC=Cp$;
and $OC+Cp=Op=OA=OB$;
$\therefore$ sq. on $CH=(OA+OF).(OB-OG)$,
$=$ rect. AF, BG.

83. Let BC be the given st. line. Divide BC in D, so that $BD : DC$ is in the given ratio. On BC describe the $\triangle$ ABC, such that $BA : AC$ is in the given ratio. Produce BC to E, so that $BE : EC = BD : DC$.

Then $\angle DAE$ is a rt. $\angle$; (p. 251, Ex. 3.)
and $\therefore$ a $\odot$ described on DE as diameter passes through A.
Let F be the centre of this circle, and G any pt. in the $\bigcirc$ce.

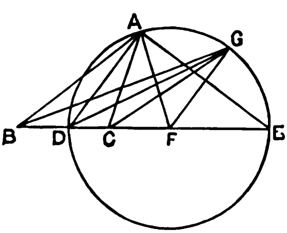


Fig. 526.

Then $\angle FAD = \angle FDA$;
$\therefore \angle FAC + \angle CAD = \angle ABD + \angle BAD$.
Now $\angle CAD = \angle BAD$; (VI. 3.)
$\therefore \angle FAC = \angle ABD$;
$\therefore \triangle$s ABF, ACF are similar;
$\therefore BF : FA = FA : FC$;
or, $BF : FG = FG : FC$;
$\therefore \angle CGF = \angle CBG$. (VI. 6.)
But $\angle FGD = \angle FDG$;
$= \angle CBG + \angle BGD$;
$\therefore \angle FGD - \angle CBG = \angle BGD$;
$\therefore \angle FGD - \angle CGF = \angle BGD$;
$\therefore \angle DGC = \angle BGD$;
$\therefore BG : GC = BD : DC$.

84. Let OA, OB be the tangents, and OCD the secant.

Since $\angle\, OAC = \angle\, ADC$, in alternate segment,

$\therefore$ $\triangle$s ADO, OAC are equiangular;

$\therefore AD : AC = DO : AO.$

Similarly $\triangle$s ODB, BCO are equiangular, and

$\therefore DB : BC = DO : OB,$

$= DO : AO;$

$\therefore AD : AC = DB : BC;$

$\therefore$ rect. AD, BC = rect. AC, DB.

(M'Dowell's *Exercises*.)

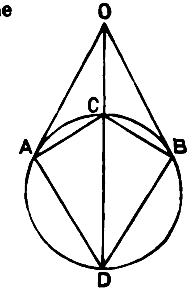


Fig. 527.

85. Let ABC, BDC be triangles on the same base, and let $\angle\, BAC = \angle\, BDC$.

Then a circle described about $\triangle$ ABC will pass through D.

Let d = diameter of this $\odot$.

Draw AM, DM $\perp$s to BC.

Then area of $\triangle$ $ABC = \frac{1}{2}\, AM \,.\, BC$;

and area of $\triangle$ $DBC = \frac{1}{2}\, DN \,.\, BC$;

$\therefore$ area of $\triangle$ ABC : area of $\triangle$ $DBC = AM : DN$,

= rect. AM, d : rect. DN, d,

= rect. BA, AC : rect. BD, DC.

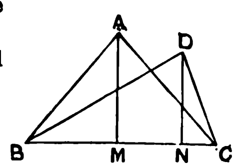


Fig. 528.

86. Let P be a pt. such that

$AP : PB = AC : CB.$

Then CP bisects $\angle\, APB$; (VI. 3.)

and if AP be produced to Q,

DP bisects $\angle\, BPQ$; (VI. A.)

$\therefore CP$ is $\perp$ to DP;

$\therefore$ a semicircle described in CD passes *through P.*

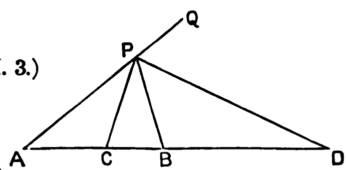


Fig. 529.

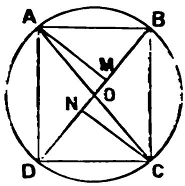


FIG. 530.

87. Let $ABCD$ be the quadrilateral, O the pt. of intersection of the diagonals.
Then if the diagonals are at rt. ∠s,

area of quadrilateral $= \frac{1}{2} AC \times BD$.

If they are not at rt. ∠s, draw $AM, CN \perp$ s to BD: then area of quadrilateral $= \frac{1}{2} (AM + CN) \times BD$.
Now AM is less than AO, and CN is less than CO;

$\therefore AM + CN$ is less than AC.

88. It is clear that if we prove the proposition for any one triangle, it will be true for all triangles having their sides parallel to the sides of the particular triangle.

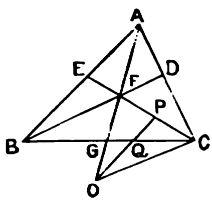


FIG. 531.

Now AF produced bisects BC in G, and $AF = 2 . FG$.

Draw $CO \parallel$ to BF, then FCO is the particular triangle to which we have referred.

Since $AD = DC$, $\therefore AF = FO$, and $\therefore GO = GF$, and the line CG bisecting FO coincides with BC, and $\therefore$ is $\parallel$ to it.

Again, if OQP bisect FC,

$CP = \frac{1}{3}$ of CE, and $CQ = \frac{2}{3}$ of $CG = \frac{1}{3}$ of CB;

$\therefore OP$ is $\parallel$ to AB.

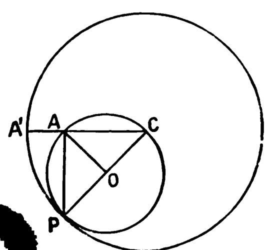


FIG. 532.

89. Let C be the centre of the larger circle, and O the centre of the smaller ⊙.

Draw CAA' to meet the larger ⊙ in A'.

Then $\angle AOP = 2 \angle A'CP$;

$\therefore$ arc $AP =$ arc $A'P$;

$\therefore A'$ is the original position of A, and PA being $\perp$ to CA' is constant in direction.

90. Join FH and EK. Then since $\angle EAK = \frac{2}{3}$ of a rt. $\angle$,

$\therefore \triangle EAK$ is equilateral, and $\therefore EK$ is $\parallel$ to CB;

and similarly FH is $\parallel$ to AC.

Then in $\triangle$s GEA, FEC, $\because \angle GEA = \angle FEC$,

and $\angle EAG$ = supplement of $\angle ECF$,

$\therefore GE : EF = GA : CF$. (1.)

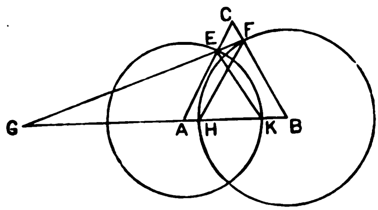


FIG. 533.

And in $\triangle$s GEK, EFC, $\because \angle GKE = \frac{2}{3}$ of a rt. $\angle = \angle ECF$,

and $\angle GEK = \angle GFB$ = supplement of $\angle EFC$,

$\therefore GE : EF = GK : CE$. (2.)

From (1.) and (2.) $GA : GK = CF : CE$.

The other result follows similarly.

91 Let OD, OE, OF be the perpendiculars on BC, AC, AB.

Join OB, OA, DF, FE.

Circles can be described about $DOEC$, $OEAF$, and $OFDB$;

$\therefore \angle BOD = \angle BFD$, and $\angle AOE = \angle AFE$.

Now $\angle DOE + \angle DCE = 2$ rt. $\angle$s,

$= \angle BOA + \angle DCE$;

$\therefore \angle DOE = \angle BOA$;

$\therefore \angle AOE = \angle BOD$;

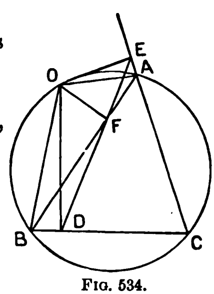


FIG. 534.

Hence $\angle BFD = \angle AFE$; and $\therefore$, since BFA is a straight line, DF and FE must be in the same straight line.

(M'DOWELL'S Exercises.)

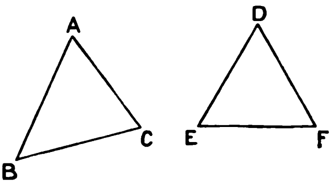


FIG. 535.

92. Let ABC, DEF be the two triangles, such that $\angle BAC = \angle EDF$, and $ED = FD$; and suppose that

$$AB : DE = DE : AC.$$

Then $AB : DE = DF : AC$;
and $\therefore$ by VI. 15,

$$\triangle ABC = \triangle DEF.$$

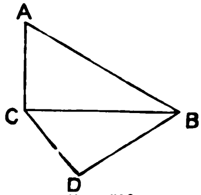


FIG. 536.

93. Since the $\triangle$s are similar,

$$AB : BC = BC : BD;$$

$\therefore AB : BD =$ duplicate ratio of $AB : BC$,

$$= \triangle ABC : \triangle BCD.$$

94. Describe any equilateral $\triangle ABC$.
Take D a point of trisection of BC, and in AD take $AO =$ the given straight line.

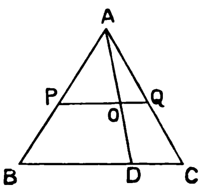


FIG. 537.

Draw $POQ \parallel$ to BC, then $\triangle APQ$ is the triangle required.

For it is equilateral, since its $\angle$s are equal to those of $\triangle ABC$, each to each, and O is a point of trisection of PQ, for

$$AD : DO = BD : PO,$$

and $AD : DO = DC : OQ$,

$$\therefore BD : DC = PO : OQ; \ \therefore PO = 2OQ.$$

95. Let d be the point through which the lines pass.

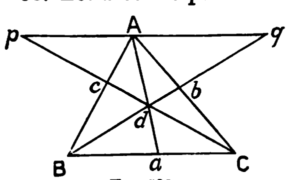


FIG. 538.

Draw $pAq \parallel$ to BC, meeting Bb, Cc produced in q and p.

Then $Aq : BC = Ab : Cb$,
and $BC : Ap = Bc : Ac$;
$\therefore Aq : Ap =$ rect. Ab, Bc : rect. Cb, Ac.
Now $Aq : Ap = Ba : Ca$;
$\therefore Ba : Ca =$ rect. Ab, Bc : rect. Cb, Ac;
$\therefore Cb : Ca =$ rect. Ab, Bc : rect. Ba, Ac.

96. Since $\triangle AFD : \triangle AFB = FD : BF$,
$= \text{rect. } AF, FD : \text{rect. } AF, BF$;
and $\triangle AFB : \triangle BFC = AF : FC$,
$= \text{rect. } AF, BF : \text{rect. } BF, FC$;
$\therefore \triangle AFD : \triangle BFC = \text{rect. } AF, FD : \text{rect. } BF, FC$.
But $\triangle AFD : \triangle BFC = \text{sq. on } AD : \text{sq. on } BC$;
$\therefore \text{rect. } AF, FD : \text{rect. } BF, FC = \text{sq. on } AD : \text{sq. on } BC$.

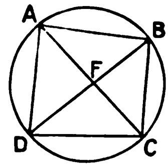


FIG. 539.

97. Since $\angle BAF = \angle BDC$, in same segment of $\odot ABED$,
and $\angle AFB = \text{supplement of } \angle BFE$,
$= \angle DCB$, by $\odot BFEC$;
$\therefore \triangle$s ABF, DBC are similar.
Again, $\angle BFC = \angle BEC = \text{supplement of } \angle BED = \angle BAD$,
and $\angle BCF = \angle BEF = \angle BDA$;
$\therefore \triangle$s BCF, ADB are similar.

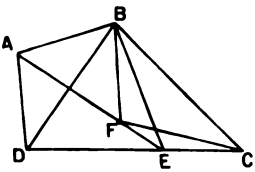


FIG. 540.

98. Draw $EF \parallel$ to AB, meeting AC in E, and BC in F.
Draw $CO, EQ, FP \parallel$ to BD.
Then O is the middle pt. of AB; (VI. 2.)
and $OP : PB = CF : FB$,
$= CE : EA$,
$= OQ : QA$;
$\therefore OP + PB : PB = OQ + QA : QA$;
$\therefore OB : PB = OA : QA$;
$\therefore PB = QA$.

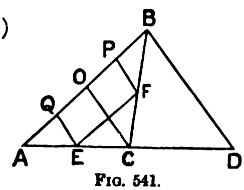


FIG. 541.

99. Since $CF : CD = EF : BD$,
and $CD : DF = AC : EF$,
$\therefore CF : DF = AC : BD$;
$\therefore CF : AC = DF : BD$;
and $\angle ACF = \angle BDF$;
$\therefore \angle AFC = \angle BFD$; (VI. 6.)
$\therefore \angle AFE = \angle BFE$.

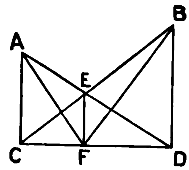


FIG. 542.

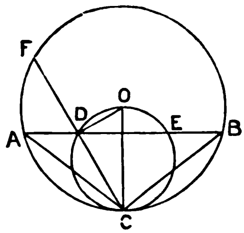


FIG. 543.

100. Let CD produced meet the larger ⊙ in F.

Then $\angle ODC$ is a rt. $\angle$, and

$\therefore DF = DC$.

Now rect. AD, DB = rect. FD, DC,

= sq. on DC.

Similarly, rect. AE, EB = sq. on CE.

101. Describe a ⊙ about the $\triangle ABC$.

Produce EB to meet the Oce in F, and join AF.

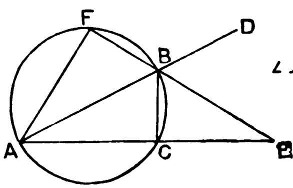


FIG. 544.

Then $\because \angle EBC = \angle EBD$,

$= \angle FBA$,

and

$\angle AFB$ = supplement of $\angle BCA = \angle BCE$,

$\therefore$ the $\triangle$s EBC, FBA are equiangular;

$\therefore AB : BF = EB : BC$;

$\therefore$ rect. AB, BC = rect. BF, EB;

$\therefore$ rect. AB, BC + sq. on BE

= rect. BF, EB + sq. on BE,

= rect. FE, EB,

= rect. EA, EC.

102. In the $\triangle$s QAB, PCB, since $\angle QBA = \angle PBC$, and $\angle QAB$ is the supplement of $\angle PCB$,

$\therefore QA : QB = PC : PB$.

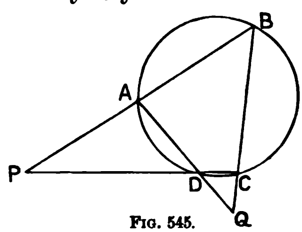


FIG. 545.

Also $\angle P + \angle Q = 4$ rt. $\angle$s $- (\angle DAB + \angle BCD + 2\angle ABC)$,

$= 2$ rt. $\angle$s $- 2\angle ABC$;

$\therefore \frac{1}{2}(\angle P + \angle Q)$ = complement of $\angle ABC$.

103. On the given diameter describe a ⊙, and from it cut off a segment ACB containing an ∠ equal to the given vertical ∠.

Divide AB in D, so that $AD : DB$ in the given ratio of the sides. Draw the diameter $EF \perp$ to AB.

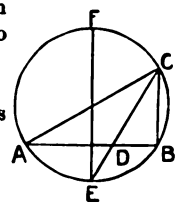


FIG. 546.

Join ED and produce it to meet the Oce in C.

Then since arc AE=arc EB, the ∠ ACB is bisected by CE;

$$\therefore AC : CB = AD : DB,$$
$$= \text{the given ratio.}$$

104. Since the distance OP is known, OC is known, and we can find a square = difference of squares on OC, CT, the side of this square gives OT.

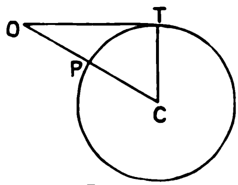


FIG. 547.

Join OC, and what is required is done.

105. Let A, B, and C be the three points.

Divide AB in D so that $AD : DB = p : q$, and in AB produced take a point E, such that $AE : BE = p : q$, and upon DE, as a diameter, describe a circle. Every point on this circle has its distances from A and B proportional to p and q respectively. (See Ex. 83.)

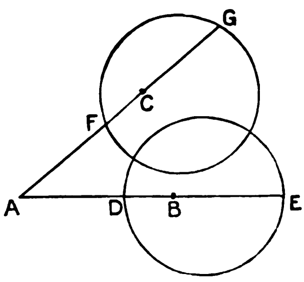


FIG. 548.

Describe a similar circle relative to A and C.

The points of intersection of these circles, when intersection is possible, satisfy the required condition.

106. Since AD is bisected at C, $\angle ABC = \angle CBD$;

$\therefore \angle ABC = \angle CAD$, and $\therefore$ a $\odot$ passing through AEB touches AC.

Draw $AA' \perp$ to CA, and $BA' \perp$ to AB.

Then AA' must be a diameter of the $\odot$ passing through AEB,

and $\angle A'AB$ = complement of $\angle CAB$,

$= \angle CBA$;

$\therefore \triangle A'AB$ is similar to $\triangle ABC$;

$\therefore BC : AB = AB : AA'$.

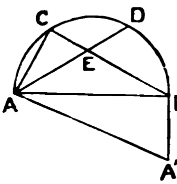


FIG. 549.

107. Draw $AT \perp$ to the given line TP, and take B in AT, such that rect. AB, AT = rect. AQ, AP.

Then $AB : AQ = AP : AT$.

Hence $\triangle$s BAQ and PAT are similar;

and $\therefore \angle AQB = \angle ATP$ = a rt. $\angle$;

$\therefore$ the locus of Q is a $\odot$ having AB for its diameter.

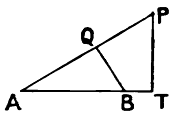


FIG. 550.

108. We can describe a rectangle having for one side the sum of the given lines, and for the other the difference. The area of this rectangle will be the difference of the squares on the lines. We can then describe on the given line a rectangle equal to this rectangle, and what was required is done.

109. Let $C'B$ and $B'C$, the external bisectors of the angles at B and C, meet in A'.

First, since $\angle A'$ = supplement of $2 \angle A$, $\angle A'$ cannot $= \angle A$ unless $\angle A = \frac{1}{3}$ of 2 rt. $\angle$s, in which case the $\triangle$s are equilateral. Hence $A'B'C'$ is not similar to ABC.

Next, to show that $A'B'C'$ cannot be similar to BCA.

If so, 2 rt. $\angle$s $- 2 \angle A = \angle B$,

and 2 rt. $\angle$s $- 2 \angle B = \angle C$,

and 2 rt. $\angle$s $- 2 \angle C = \angle A$.

Hence we get $\angle A - \angle C = O$,

and $\angle A - \angle B = O$.

$\therefore$ the $\triangle$s are equilateral.

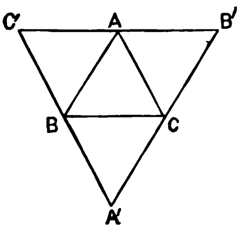


FIG. 551.

110. Area of ABC : area of $A'B'C'$
$=$ rect. AB, AC : rect. $A'B', A'C'$,
$=$ rect. $A'C', AC$: rect. $A'B', A'C'$,
$= AC : A'B'$.

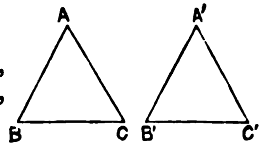


Fig. 552.

111. Let A, B, C, D, E, F be the angular pts. of a regular hexagon. Describe a ⊙ about it.

Join AE, EC, CA; BF, FD, DB.

Then △s AEC, BDF are equilateral.

And since FB, EC cut off equal arcs, they are ||.

∴ $AM = AN$, and △ AMN is equilateral.

Now in △s ABM, FEM,

∵ $AB = FE$, and ∠ AMB = ∠ FME,
and ∠ ABM = ∠ FEM, in the same segment,

∴ $AM = FM$; and similarly $AN = BN$;

∴ $FM = MN = NB$;

∴ △ $FRM = \frac{1}{3}$ of △ BDF = △ BON = △ QPD;

∴ hexagon $MNOPQR = \frac{2}{3}$ of △ BDF,
$= \frac{2}{3}$ of $\frac{1}{2}$ of hexagon $ABCDEF$, (p. 202, IV. 15.)
$= \frac{1}{3}$ of hexagon $ABCDEF$.

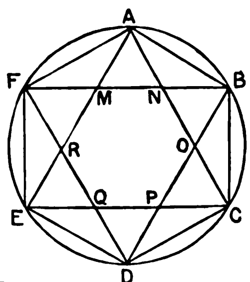


Fig. 553.

112. Let ABC be the △, and DE the line || to BC.

Then △ $ABC = 9$ times △ ADE;

∴ AC = three times AE;

∴ D, E are points of trisection of AB, AC.

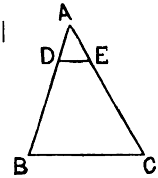


Fig. 554.

113. Let APQ be the chord to the circles from A. Join CP, and draw DR to R the middle pt. of PQ. Then CP, DR being both $\perp$ to AQ are $\parallel$.

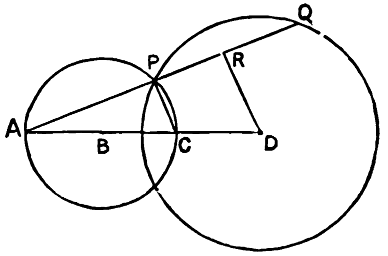


FIG. 555.

$$\therefore AP : PR = AC : CD;$$
$$\therefore AP = 2\ PR;$$
$$\therefore AP = PQ.$$

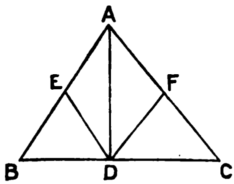


FIG. 556.

114. Since $EA = EB = ED$, a $\odot$ described with centre E and radius EA passes through D and B;

$\therefore \angle ADB$ is a rt. $\angle$;

$\therefore ADC$ is a rt. $\angle$;

$\therefore$ a $\odot$ described with centre F and radius FA passes through D.

Next, let AEF be an acute-angled $\triangle$. Produce AE, AF to B and C, so that $EB = AE$, and $FC = AF$.

Then BC is $\parallel$ to EF.

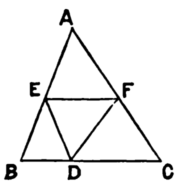


FIG. 557.

Turn $\triangle AEF$ over round EF, then A will fall on a pt. D in BC such that $ED = EA$, and $FD = FA$; and $\therefore$ $\triangle EAF$ will coincide with $\triangle EDF$.

Then $\angle EAF + \angle AEF + \angle AFE$
$= \angle EDF + \angle FED + \angle EFD,$
$= \angle EDF + \angle EDB + \angle FDC,$
$= 2$ rt. $\angle$s.

115. Since $\triangle BAC = \triangle DAE$,

$\therefore \triangle BCE = \triangle DCE$;

and $\therefore$ CE is $\parallel$ to BD;

$\therefore BC : BF = DE : DF$;

and $\therefore$ $\triangle BCA : \triangle BFA = \triangle DEA : \triangle DFA$.

But $\triangle BCA = \triangle DEA$;

$\therefore \triangle BFA = \triangle DFA$;

and $\therefore$ perpendiculars from B and D on FA are equal;

and $\therefore$ FA bisects BD and its parallel CE.

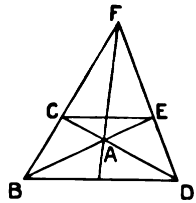


FIG. 558.

116. Sq. on PN = rect. AN, NB,

$= \frac{1}{8} \cdot \frac{5}{8}$ of sq. on AB;

sq. on QN = rect. AN, NO,

$= \frac{1}{8} \cdot \frac{3}{8}$ of sq. on AB,

$\therefore$ sq. on PN : sq. on QN = 5 : 2.

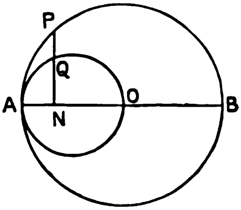


FIG. 559.

117. First describe a rectangle equal to the sum of the given square and the given rectangle. Then describe a square equal to this rectangle.

Miscellaneous Exercises on Book XI.

Page 334.

1. Let OA, OB, straight lines in the plane AOB, be equally inclined to the plane ACB.

Let AB be the common section of the planes. From O draw $OC \perp$ to the plane ACB, and join AC, BC.

Then $\because \angle OAC = \angle OBC$, by hypothesis, and $\angle OCB = \angle OCA$, each being a rt. $\angle$, and OC is common to the $\triangle$s AOC, BOC,

$\therefore OA = OB$,

and $\therefore \angle OAB = \angle OBA$.

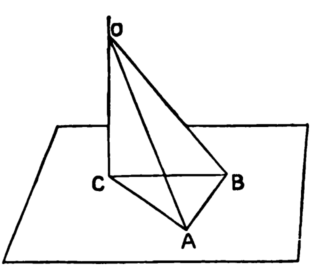


FIG. 560.

2. The two planes $ACBX$, $ACBY$ are at rt. $\angle$s.

In plane $ACBX$ from the pt. C in the intersection AB of the planes, CE, CF are drawn making $\angle ACE = \angle BCF$. DCD' is any straight line drawn through C in the plane $ACBY$. Take $CE = CF$.

Draw EA, $FB \perp$ to AB.

Then $\because CE = CF$, and $\angle CAE = \angle CBF$, and $\angle ECA = \angle BCF$,

$\therefore AC = BC$, and $AE = BF$.

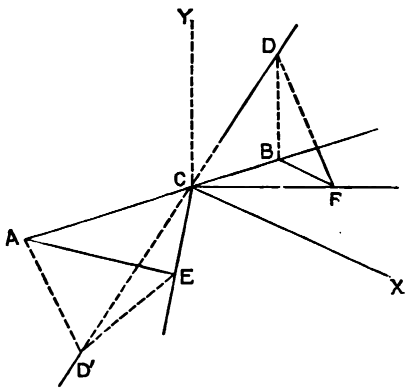


FIG. 561.

Again, draw in plane $ACBY$ the lines AD', $BD \perp$ to AB, and meeting DCD' in D' and D.

Then $\because AC = BC$, and $\angle ACD' = \angle BCD$, and $\angle CAD' = \angle CBD$,

$\therefore CD' = CD$, and $AD' = BD$.

Join $D'E$, DF, and then

$\because AD' = BD$, and $AE = BF$, and $\angle D'AE = \angle DBF$,

$\therefore D'E = DF$; and $\therefore$ in the $\triangle$s CED', CFD,

$\because CD' = CD$, and $CE = CF$, and $D'E = DF$,

$\therefore \angle D'CE = \angle DCF$.

3. Draw $AF \perp$ to BC, and $BG \perp$ to AC, and let these $\perp$s intersect in D. Join EF.

Then $\because DE$ is $\perp$ to plane ABC, every plane through DE is $\perp$ to

plane ABC, and $\therefore$ the plane through DE and AF is $\perp$ to plane ABC, and AF is the intersection of the planes. Now FB is drawn in

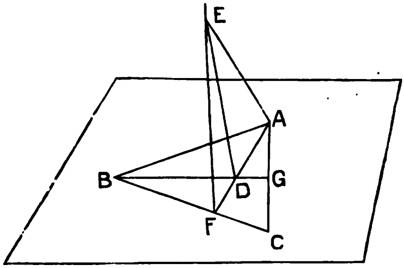


FIG. 562.

plane ABC, and is $\perp$ to the section AF, and $\therefore$ is $\perp$ to every line drawn in the plane EAF, and EA is in this plane, and $\therefore$ BC is $\perp$ to EA. Similarly, BE is $\perp$ to AC, and EC is $\perp$ to AB.

4. Let AB be the common intersection of a number of planes $EABE'$, $FABF'$, $GABG'$ . . . and D the given point. Through D draw a plane $DCefg \perp$ to the section AB, and cutting it in C; then since ACB is $\perp$ to the plane $DCefg$, every plane through ACB is $\perp$ to $DCefg$, and $\therefore$ $DCefg$ is $\perp$ to all the given planes having the common section AB, and $\therefore$, if De, Df, Dg . . . be $\perp$s from D on these planes, e, f, g . . . all lie in the same plane through $DC \perp$ to AB, and $\therefore$ since CD is fixed, and the $\angle$s made by joining C to e, f, g . . . are all rt. $\angle$s, the locus of feet of $\perp$s is a circle on DC as diameter.

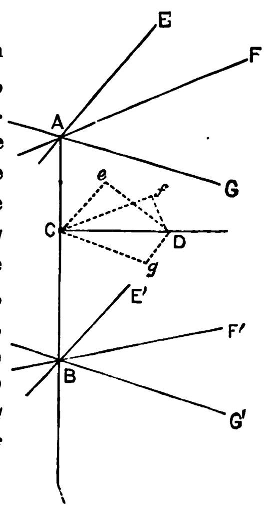


FIG. 563.

5. Let $EABE'$, $FABF'$ (as in diagram to Ex. 4) be the given planes. Let D be the pt. from which the $\perp$s on these planes are drawn. Through D draw a plane $DCef \perp$ to AB, the section of the given planes, and draw De, Df $\perp$s from D on these planes.

Then $\angle eCf$ = angle of inclination of planes $EABE'$, $FABF'$.

And since De is $\perp$ to Ce, and Df is $\perp$ to Cf,

$$\therefore \angle eDf = \angle eCf.$$

6. Let $AH'GH$ be the plane through $A \perp$ to plane $AFF'A'$, having the line HGH' its intersection with the plane $HFH'F'$.

Now since AF, $A'F'$ are two $\perp$s to the plane $HFH'F'$, this plane is $\perp$ to the plane drawn through AF, $A'F'$, and also plane $AH'GH$ is $\perp$ to the plane through AF, $A'F'$.

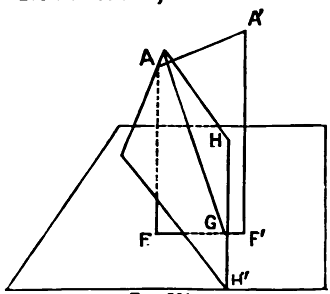


Fig. 564.

$\therefore$ the two planes $FHH'F'$, $AH'GH$, intersecting in the line HGH', are each of them $\perp$ to the plane $AFF'A'$, and $\therefore$ the line HGH' is $\perp$ to the plane $AFF'A'$, and $\therefore \perp$ to every straight line drawn through G in that plane, and $\therefore \perp$ to FGF'.

7. Let FB, FC be equal straight lines drawn from the pt. F to the plane ECB. Draw $FE \perp$ to the plane ECB, and join EB, EC.

Then FE is $\perp$ to EB and also to EC.

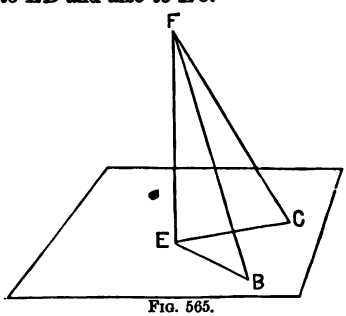


Fig. 565.

$\therefore$ in the right-angled $\triangle$s FEB, FEC, $FB = FC$, and FE is common, and $\therefore \angle FBE = \angle FCE$, that is, FB and FC are equally inclined to *the plane.*

8. This exercise is the same as Ex. 5.

9. From A draw $AB \perp$ to the plane BDE, and $AD \perp$ to the line DE in that plane.

Make $DE = AB$, and join AE, BE, BD.

Then in $\triangle$s ABE, ADE,

$\because$ $AB = DE$, and AE is common, and rt. $\angle\, ABE =$ rt. $\angle\, ADE$;

$\therefore$ $AD = BE$.

Then in $\triangle$s ABD, BDE,

$\because$ $AD = BE$, and BD is common, and $AB = ED$;

$\therefore$ $\angle\, ABD = \angle\, BDE$;

$\therefore$ $\angle\, BDE$ is a right angle.

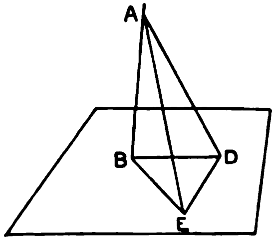


FIG. 566.

10. The angles containing a solid angle are together less than four rt. $\angle$ s.

$\therefore$ if equilateral $\triangle$s alone be used, a solid angle can be made by 5, 4, or 3 equilateral triangles. If 6 were employed, the solid $\angle$ would become equal to four rt. $\angle$s, and $\therefore$ the bounding lines would all be in one plane. If only squares be used, there is only one way of forming a solid $\angle$, that is by using 3, for if 4 were used, the solid $\angle$ would become equal to four rt. $\angle$s. Taking *one* square, we can form a solid $\angle$ either with 4, 3, or 2 equilateral $\triangle$s.

Taking *two* squares, we can form a solid $\angle$ with 2 equilateral $\triangle$s only, for if 3 were used, the solid $\angle$ would become equal to four rt. $\angle$ s.

$\therefore$ the total number of ways is $3 + 1 + 3 + 1$, or, 8.

11. Let AB be the intersection of the two planes, inclined at a given $\angle$.

Let BCD be a plane drawn $\perp$ to AB, and having BC, BD for its intersections with the given planes. Join any two pts. C and D, one in each of these intersections. With CD as diameter describe a sphere cutting AB in E, and join CE, DE. Then the plane through C, D, E intersects the two given planes in CE, DE, and these lines are at right angles, since CD is the diameter of a sphere, and E is a point in the circumference of the sphere.

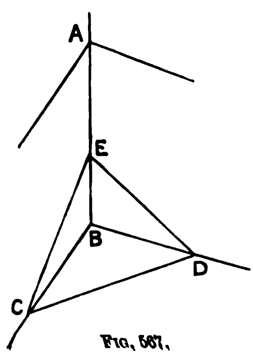


FIG. 567.

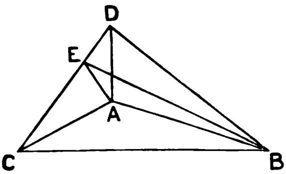


Fig. 568.

12. Since AB is $\perp$ to AD and also to AC, it is $\perp$ to the plane DAC, and $\therefore$ to every st. line in that plane, and $\therefore$ to AE and DE.

But, by construction, AE is $\perp$ to CD, and $\therefore$ DE is $\perp$ to AE and also to AB, and $\therefore$ to the plane passing through AE, AB, and $\therefore$ to every st. line in this plane, and $\therefore$ to EB.

13. Let A be a pt. in the wall XOO', and B a pt. in the wall YOO'. The shortest distance from A to B measured along the planes is ACB, where AC, CB make equal angles with $O'O''$: for if we imagine the plane YOB to be turned about the intersection OO' till it coincides with the plane XOA, then ACB will be a st. line, and $\therefore$ the shortest distance between A and B. To find the pt. C, draw AO'' in the plane XOO', $\perp$ to OO', and BO' in the plane YOO' $\perp$ to OO'. Divide $O'O''$ in C so that $O''C : O'C = AO'' : BO'$. Then since $\triangle$s $AO''C$, $BO'C$ have $\angle$s at O'', O' rt. $\angle$s, and the sides about them proportional,

$$\therefore \angle ACO'' = \angle BCO';$$

$\therefore$ C the required pt. is found.

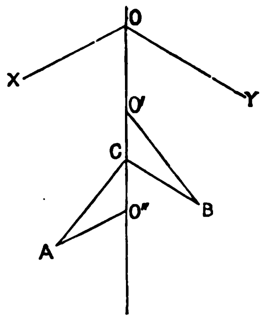


Fig. 569.

14. Let FDG be the given plane, A and B the given pts. Join AB, and through A and B draw the $\perp$s AD, BE to the plane FDG. If C be the pt. in FDG in which the lines from A and B intersect when the distance is the shortest possible, it must lie in the plane through $AB \perp$ to FDG, and $\therefore$ in the line DE. For if not, suppose C' to be the point; and draw $C'C$ $\perp$ to DE in plane FDG, and join AC', BC'. Then we have the right-angled $\triangle$s ACC', BCC', and the sum of the hypotenuses AC', BC' must be greater than the sum of AC, BC; and $\therefore$ C' cannot be the point, and $\therefore$ the point must be in DE.

Again, if we produce AD to A', so that $A'D = AD$, it is plain that A' is at the same distance from any pt. C in the plane FDG as A is, and $\therefore$ if we join BA', BA' is the shortest distance between B and A', and if BA' meets the plane FGD in C, BC, CA' together equal BC, CA together, and C is the required pt. But BCA' is a straight line, and

$\therefore \angle BCE = \angle DCA' = \angle ACD$.

Hence the sum of the two st. lines is the least possible when they are drawn to the intersection of the plane through A and B $\perp$ to the given plane, and make equal angles with it.

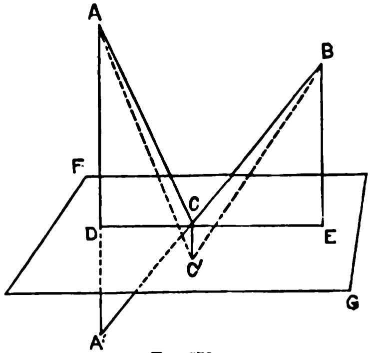


FIG. 570.

15. The two planes ABC, abc, are cut first by a plane $ABba$, in AB, ba, and next by a plane $ACca$, in AC, ca.

By XI. 16, AB is parallel to ab,

and AC is parallel to ac,

and since AB, AC, two st. lines that meet, are respectively parallel to ab, ac, two other st. lines that meet,

$\therefore$ by XI. 10, $\angle BAC = \angle bac$.

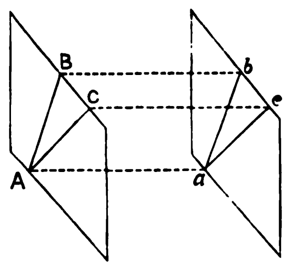


FIG. 571.

16. In the $\triangle$s BDA, BCA the two sides BD, DA are by hypothesis equal respectively to AC, CB, and the base AB is common,

$$\therefore \angle BDA = \angle ACB.$$

Similarly,

$\angle ADC = \angle ABC$, and $\angle CDB = \angle CAB$;

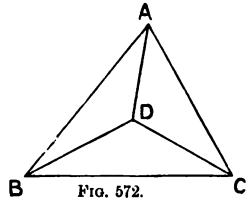


FIG. 572.

$$\therefore \angle BDA + \angle ADC + \angle CDB = \angle ACB + \angle ABC + \angle CAB,$$
$$= 2 \text{ rt. } \angle\text{s}.$$

17. Let OA, OB, OC be the three st. lines.

Take $OA = OB = OC$, and join AB, BC, CA, which lie in the plane ABC.

In the plane ABC find D, the centre of the circle circumscribing the $\triangle ABC$, and join AD, BD, CD, OD.

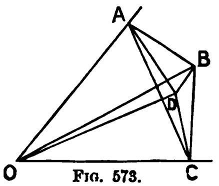


FIG. 573.

Then shall OD be the line required.

For in $\triangle$s AOD, BOD,

$\because AO = BO$, and OD is common, and $AD = BD$ (each being a radius of the $\odot$ described about $\triangle ABC$),

$$\therefore \angle AOD = \angle BOD.$$

Similarly, $\angle AOD = \angle COD$,

$\therefore$ the line drawn from O to the centre of the $\odot$ described about $\triangle ABC$ is the line required.

18. Let $ABCD$ be a triangular pyramid standing on the equilateral base BCD, and having the angles at A rt. angles.

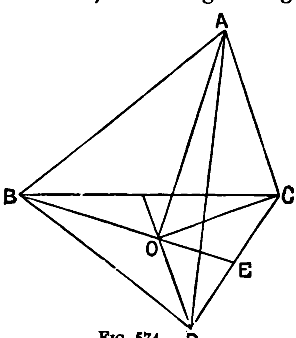


FIG. 574.

In $\triangle$s ADB, ADC,

$\because DB = DC$, and AD is common, and $\angle DAB = \angle DAC$,

$$\therefore AB = AC.$$

Similarly, $AB = AD$.

Then if O be the point of intersection of $\perp$s from the angular points of the $\triangle BDC$ on the opposite sides,

$\because OD = OB$, and OA is common, and $AD = AB$,

$$\therefore \angle AOD = \angle AOB;$$

and similarly, $\angle AOD = \angle AOC$.

Hence AO is $\perp$ to plane BDC.

Then sq. on AO = sq. on AD − sq. on DO,

= sq. on $AD - \frac{1}{3}$ sq. on DB,

= sq on $AD - \frac{2}{3}$ sq. on AD,

= $\frac{1}{3}$ sq. on AD.

19. The three plane angles BOA, COA, COB form a solid angle at O, COB being a rt. $\angle$, and COA the supplement of BOA, which we will take as acute. ABC is a plane cutting the edges OA, OB, OC in A, B, C, so that $OB = OC$.

Then in $\triangle AOB$

sq. on AB = sq. on OA + sq. on OB − 2 rect. contained by OB and the section of OB between O and foot of $\perp$ from A on OB;

sq. on AC = sq. on OA + sq. on OC + 2 rect. contained by OC and the section of OC between O and foot of $\perp$ from A on OC;

sq. on BC = sq. on BO + sq. on OC;

$\therefore$ sq. on AB + sq. on AC + sq. on BC = 2 (sq. on OA + sq. on OB + sq. on OC).

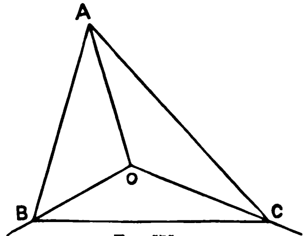


Fig. 575.

20. In the solid $\angle$ formed by OP, OA, OB
$\angle POA + \angle POB$ is greater than $\angle AOB$;
and in the solid $\angle$ formed by OP, OB, OC,
$\angle POB + \angle POC$ is greater than $\angle BOC$;
and in the solid $\angle$ formed by OP, OC, OA,
$\angle POC + \angle POA$ is greater than $\angle COA$;
$\therefore \angle POA + \angle POB + \angle POC$ is greater than $\frac{1}{2}(\angle AOB + \angle BOC + \angle COA)$.

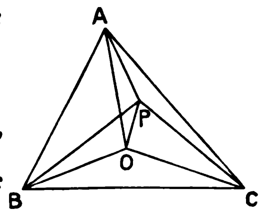


Fig. 576.

Page 342.

Senate-House Riders on Books VI. XI. and XII.

1849. VI. 4. Let AOD be a chord passing through the fixed point O in a circle. Draw any other chord BOC. Join AB, CD. Then $\triangle$s AOB, COD are similar;

$\therefore AO : OB = CO : OD$;

and $\therefore$ rect. AO, OD = rect. OB, OC;

that is, the rectangle under the segments of any *chord passing through* O is constant.

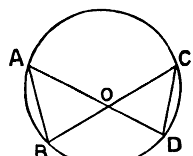


Fig. 577.

XI. 11. This has been proved in Ex. 7 on page 334.

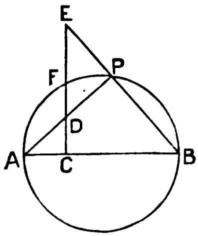


Fig. 578.

1850. VI, 10. Since the ∠s at P are rt. ∠s,

∴ △s ADC, EDP are similar;

and ∴ △s ECB, ACD are similar;

$\therefore AC : CD = EC : CB$;

∴ rect. CD, EC = rect. AC, CB,

= sq. on CF;

$\therefore CE : CF = CF : CD$.

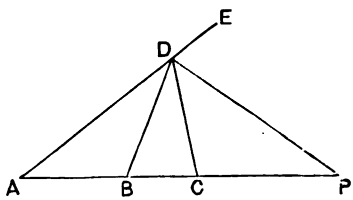


Fig. 579.

1851. VI. 3. Produce AD to E, and draw DP bisecting the ∠ CDE. Then

$\angle PDB = \frac{1}{2}(\angle ADC + \angle CDE)$.

$\therefore \angle PDB =$ a rt. ∠,

∴ a ⊙ described on BP as diameter passes through D.

(See also Ex. 83 on p. 302.)

XI. 8. Through E draw the plane CAD ⊥ to AB, intersecting the planes in AC, AD, and in the plane $ACED$ draw EC, ED ⊥s to AC, AD.

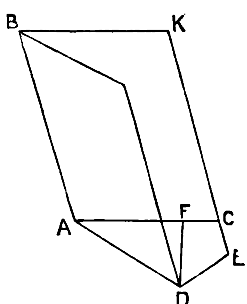


Fig. 580.

Through C draw, in the plane BAC, $CK \parallel$ to AB.

Then ECK is a rt. ∠, and also ACE is a rt. ∠.

∴ EC is ⊥ to plane BAC.

Similarly, ED is ⊥ to plane BAD.

Draw DF ⊥ to AC in plane CAD.

Then by drawing from F in plane BAC a line ∥ to AB, it may be shown that DF is ⊥ to plane BAC.

But F is a point in the line AC, and ∴ CF produced is ⊥ to AB.

1852. VI. 2. This has been proved in page 294, Ex. 12.

XI. 11. O is the point in which the perpendiculars from the angular points of BCD on the opposite sides intersect.

(See Ex. 18, p. 335.)

Then sq. on AD = sq. on AO + sq. on DO,

= sq. on AO + $\frac{1}{3}$ sq. on BD;

∴ 3 sq. on AD = 3 sq. on AO + sq. on AD;

∴ 2 sq. on AD = 3 sq. on AO;

∴ 2 sq. on AB = 3 sq. on AO.

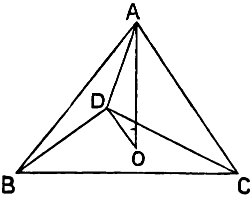


Fig. 581.

1853. VI. 6. Describe a ⊙ round ABC; then CD cuts arc AB in E' its middle point, and if the base AB and the vertical angle at C be given, E' is a fixed point.

From CA measure $CF = CB$; join FD, FE', $E'B$.

Then ∵ $CF = CB$, CD is common, and

$\angle FCD = \angle BCD$,

∴ $DF = DB$, and similarly $FE' = BE'$.

Hence $\angle DE'F = \angle DE'B$,

$= \angle DAF$;

∴ a ⊙ can be described about $FAE'D$;

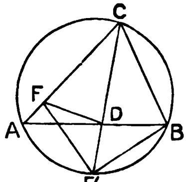


Fig. 582.

∴ rect. CD, CE' = rect. CA, CF,

= rect. CA, CB,

= rect. CD, CE (by the question)

∴ E coincides with E', a fixed point.

XI. 21. In the diagram A', B, C, A are supposed to be in the plane of the paper, and D above it.

Then since BA, CA are respectively ⊥ to the planes $BA'D$, $CA'D$, it follows that $A'D$, the intersection of the planes, is ⊥ to BA and CA, and therefore is ⊥ to the plane containing them, that is $BA'C$.

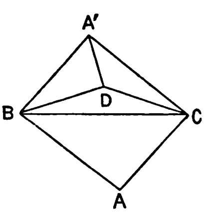


Fig. 583.

∴ $DA'B$ and $DA'C$ are rt. ∠s, and ∠ $BA'C$ is the supplement of ∠ BAC,

∴ ∠s at A' together with ∠ BAC make four rt. ∠s.

1854. VI. 16. Let O, O' be the respective centres; join CO, $C'O'$, BE, $B'E$.

Then since BE is || to $O'C'$;

$$\therefore AB : BC' = AE : EO';$$

$$\text{or } AB : BC' = AE : \tfrac{1}{2} A'E.$$

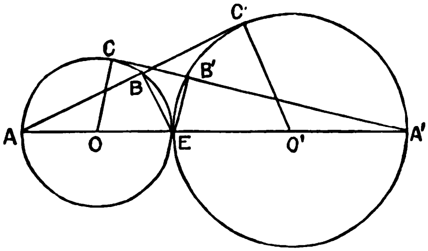


Fig. 584.

Similarly, $A'B' : B'C = A'E : \tfrac{1}{2} AE$.

Compounding, $AB, A'B' : BC', B'C = AE, A'E : \tfrac{1}{4} . A'E, AE,$

$$= 4 : 1;$$

$$\therefore \text{rect. } AB, A'B' = 4 \text{ rect. } BC', B'C.$$

XI. 20. The inner triangle need not have its sides parallel to those of the outer. Let abc be the inner and ABC the outer. Let ab, bc, ca be produced to meet the sides of ABC in γ, a, β. Join bC, and let O be the point not in the plane of the triangles; and let any line, such as ac, stand for the angle subtended by it at O.

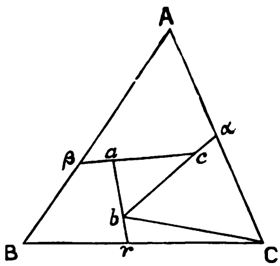


Fig. 585.

Then $b\gamma + \gamma C$ is greater than bC; (XI. 20.)

$\therefore b\gamma + \gamma C + Ca$ is greater than $bC + Ca$;

$\therefore b\gamma + \gamma C + Ca$ is greater than ba;

$\therefore b\gamma + \gamma C + Ca$ is greater than $bc + ca$;

so also, $ca + aA + A\beta$ is greater than $ca + a\beta$;

and, $a\beta + \beta B + B\gamma$ is greater than $ab + b\gamma$;

$\therefore$, adding, and removing equal lines from each side,

$\gamma C + Ca + aA + A\beta + \beta B + B\gamma$ is greater than $bc + ca + ab$;

that is, $CA + AB + BC$ is greater than $bc + ca + ab$,

$\therefore$ the particular case is also proved.

1855. VI. 2. Since BD and EC are parallel,

$\therefore$ $\triangle$s BFD, EFC are similar,

$\therefore FB : FE = BD : EC.$

Now $BD = BA$, and $EC = CA$,

$\therefore FB : FE = BA : CA,$

$\therefore$ AF is parallel to BD and EC.

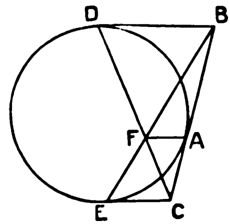


FIG. 586.

XI. 16. The planes aB, cD are evidently parallel;

$\therefore$, since they are cut by the plane ad, ab and cd are parallel.

From a draw aE parallel to AB; aE will evidently lie in the plane aB, and will intersect Bb in some point E, so that $aEBA$ shall be a parallelogram, and $\therefore aE = AB$.

Similarly, draw $cF \parallel$ to CD to intersect Dd in F, then, as before, $cF = CD$.

Now, since ab is $\parallel$ to cd, and aE is $\parallel$ to AB, which is $\parallel$ to CD, which is $\parallel$ to cF, $\therefore aE$ is $\parallel$ to cF.

Hence $\angle baE = \angle dcF$.

Similarly, $\angle abE = \angle cdF$,

$\therefore$ $\triangle$s abE, cdF are similar;

$\therefore ba : aE = cd : cF;$

and $\therefore ba : AB = cd : CD.$ (I. 34.)

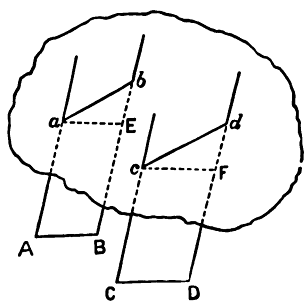


FIG. 587.

1856. XI. 11. Let $ABCD$ be the regular tetrahedron.

Bisect CD in E. Join AE, BE, and take F in BE such that $BF = 2\ FE$. Join AF.

Now F is the pt. where the $\perp$ from A on BCD meets BCD (see p. 335, Ex. 18). But all such $\perp$ s in a regular tetrahedron are equal; and hence if BG be the $\perp$ from B on ACD, $AF = BG$.

Draw $FH \perp$ to AE. Then by similar $\triangle$s BGE, FHE,

$$BG : FH = BE : FE;$$
$$= 3 : 1.$$

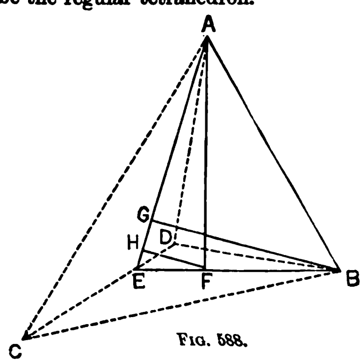


FIG. 588.

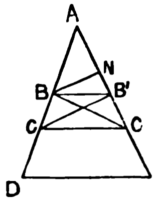


FIG. 589.

1857. VI. 19. The triangle $AC'B$ always varies as its base and height jointly. Draw $BN \perp$ to AB', then $AC'B$ varies as the rectangle AC', BN.

Suppose $B_{,}B_{,}'$, $C_{,}C_{,}'$ to be new positions of BB' and CC': then ($N_{,}$ being the new position of N) the triangle $AC_{,}'B_{,}$ would vary as the rectangle $AC_{,}'$, $B_{,}N_{,}$.

But the $\triangle$s BNB', $B_{,}N_{,}B_{,}'$ are similar;

$$\therefore \triangle AC'B : \triangle AC_{,}'B_{,} = \text{rect. } AC', BN : \text{rect. } AC_{,}', B_{,}N_{,}$$
$$= \text{rect. } AC', B'B : \text{rect. } AC_{,}', B_{,}'B_{,}$$
$$= \text{rect. } CC', B'B : \text{rect. } C_{,}C_{,}', B_{,}'B_{,}$$

that is, $\triangle AC'B$ varies as rect. CC', BB'.

Similarly, the proposition may be shown to be true for the $\triangle AB'C$.

XI. 16. Let $OABC$ be the pyramid, ABC the equilateral base.

Bisect AB in F, join CF, OF. CF is evidently $\perp$ to AB, and $AB \perp$ to plane OCF, and $\therefore$ to OF. But F is the middle pt. of AB, and $\therefore$ $\triangle$s OFB, OFA are equal in every respect;

$$\therefore OB = OA \text{; and, by similar reasoning, } OB = OC.$$

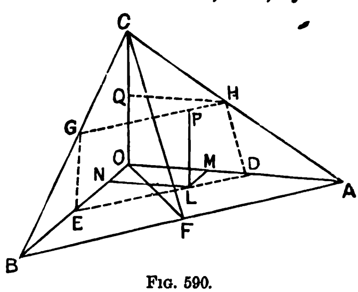


FIG. 590.

Take any pt. P in the plane ABC; draw $PL \perp$ to OAB, through PL draw the plane $EPD \parallel$ to AB and cutting OAB in ED, and CAB in GH (where evidently, by construction, both ED and GH will be $\parallel$ to AB). The $\perp$ s from P on the planes OAC, OBC are evidently respectively equal to LM and LN.

For any position of P along GH, the $\perp$ on the plane OAB, will always be equal to PL. Hence for positions of P along GH, the sum of the $\perp$ s will always be the same, if the sum of LM and LN be the *same.*

In Fig. 591, let L', M', N' be new positions of L, M, N; $L'M', LN$ intersecting in K, then evidently $LM + LN = L'M' + L'N'$, if $L'K = LK$, which is true, because $L'KL$ is a $\triangle$ similar to BOA, and $\therefore$ isosceles. Hence for any pt. P in GH the sum of the $\perp$s is equal to $PL + LM + LN$, that is, to $PL + DO$, or $HD + DO$, or $HD + HQ$ (HQ being $\parallel$ to DO).

Exactly as before, $HD + HQ$ is of constant value and equal to AO.

But P was taken anywhere in ABC, and the sum of its $\perp$s has been found to be equal to AO, and $\therefore$ the sum of the $\perp$s from any point in the $\triangle ABC$ (and *within* it) is constant and equal to AO (or BO or CO).

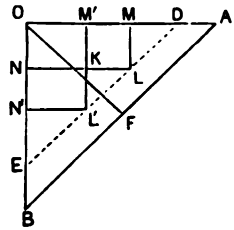


FIG. 591.

The same is true if P be in the plane ABC and *without* the $\triangle ABC$, provided that the $\perp$s be subtracted when they fall on the side of the $\triangle$s OAB, OBC, OCA opposite to that on which they fell when P is *within* the $\triangle ABC$.

1858. VI. 15. Let ABC be the $\triangle$, and BC its base.

Take P in AB such that

sq. on AP = rect. AB, BP. (II. 11.)

Draw $PD \parallel$ to BC, to meet AC in D, and join PC.

Then shall P be the point required.

For since sq. on AP = rect. AB, BP,

$\therefore AP : PB = AB : AP$;

and $\therefore$, since $\triangle$s APD, ABC are similar,

$AP : PB = BC : PD$;

and $\angle APD = \angle PBC$;

$\therefore$ (by VI. 15) $\triangle APD = \triangle PBC$.

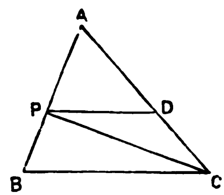


FIG. 592.

XI. 11. Cut the given planes by a plane $\perp$ to their line of intersection. Let the given planes cut this plane in the lines XOX', YOY'. Each required locus will reduce to a point where it cuts the above plane of construction (the plane of the paper). P, Q, R, S are these *points*.

Bisect $\angle$ YOX by OP, and from O draw $ON \perp$ to YO, and equal to the given line in length; draw $NP \parallel$ to OY to cut OP in P; and draw PM, PL $\perp$s to OY, OX. The $\triangle$s POM, POL are evidently equal, and $\therefore$ $PM = PL =$ given line.

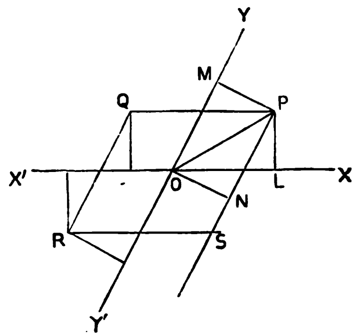


FIG. 593.

Construct similarly for R. Through P and R draw PQ, RS $\parallel$ to OX, and PS, RQ $\parallel$ to OY, intersecting in Q and S.

Then P, Q, R, S are evidently the required points.

Moreover, PQ, QR, RS, SP are lines of intersection of the four planes (of the second part of the rider) with the plane of the paper. Perpendiculars from O on these lines (ON is one of them) are evidently all equal to the given line.

1859. VI. 31. By VI. 18 this rider is self-evident: it being only necessary to notice that if two or more sides of the given polygon are equal, an equal number of the described polygons will be equal, so that there will be only as many polygons of *different* sizes as there are sides of *different* sizes in the original polygon.

XI. 20. Let the four lines be cut by a plane in the points A, B, C, X, the lines being OA, OB, OC, OX. Join AB, BC, CA, AX, BX, CX; and let the $\angle$s subtended at O by AB, BC, CA, AX, BX, CX be represented respectively by c, a, b, x, y, z.

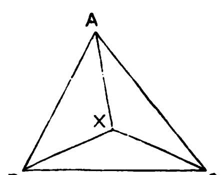


FIG. 594.

Then (see solution of rider for 1869, XI. 20)

$c + b$ is greater than $y + z$,
$a + c$ is greater than $x + z$,
$b + a$ is greater than $y + x$;
$\therefore$ $a + b + c$ is greater than $x + y + z$.

Again, in the pyramid $OABX$, by XI. 20,

$x + y$ is greater than c,
so also, $y + z$ is greater than a,
and $z + x$ is greater than b;
$\therefore$ $2(x + y + z)$ is greater than $a + b + c$;
$\therefore$ $x + y + z$ is greater than $\frac{1}{2}(a + b + c)$.

1860. VI. A.—I. The proposition says that when $\angle DCB$ is bisected by CX, $AX : BX = AC : BC$.

Now in the mind suppose CX always to bisect $\angle DCB$, while the triangle changes itself gradually to suit X moving away from A and B indefinitely, while C's position is always the same. Then evidently CX will come to fulfil Euclid's definition of parallelism with regard to AB. When this is the case,

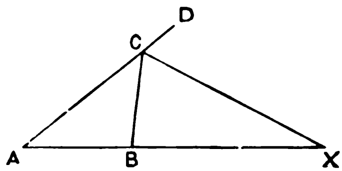


FIG. 595.

$$\angle DCX = \angle CAX, \text{ and } \angle DCX = \angle BCX = \angle CBA;$$
$$\therefore \angle CAX = \angle CBA, \text{ or } CA = CB.$$

Hence when $CA = CB$ the external bisector CX is $\parallel$ to base AB.

II. Taking an evident construction, since CG bisects $\angle ACB$,
$$\therefore AG : GB = AC : CB;$$
$$\text{or, } AG : AC = GB : CB.$$

Let the bisector AF of the external $\angle BAD$ meet CG produced in F; then
$$FG : FC = AG : AC,$$
$$= GB : CB;$$

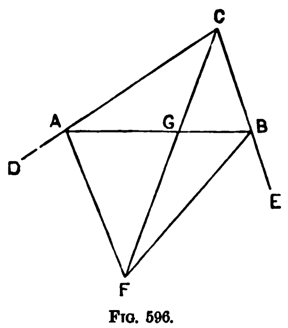


FIG. 596.

$\therefore$ joining FB, by VI. A. FB is the bisector of external $\angle ABE$, that is, the external bisectors of A and B meet the internal bisector of C *in the same point.*

XI. 17. Let ACE, GHK, BDF be the three planes, of which ACE and GHK are parallel; AGB, CHD, EKF the three lines cut by these planes, so that $AG : CH : EK = GB : HD : KF$.

From C draw $CLMN \parallel$ to AGB, cutting the three planes in C, L, N; join AC, GL, BN; LH, ND.

Now AL is a parallelogram; and so also is GN, if the plane BDF is $\parallel$ to GHK, and in this case the rider is proved by XI. 17.

If this is not so, take LM (in LN) $= GB$, and join BM, MD. Then GM is a ▱;

$$\therefore AG : GB = CL : LM;$$
$$\therefore CH : HD = CL : LM;$$
$$\text{and } \therefore LH \text{ is } \parallel \text{ to } MD;$$

$\therefore$ BDM is a plane $\parallel$ to GHK, and different from the plane BDF.

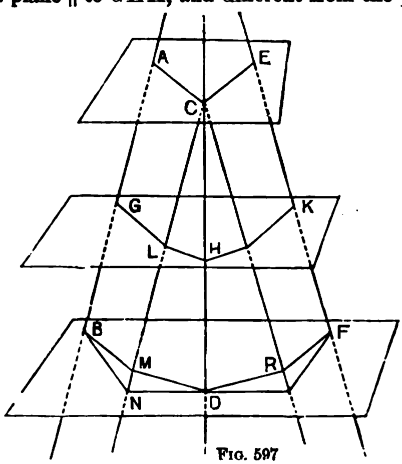


Fig. 597

In a similar manner another plane DRF may be found passing through D and F, and $\parallel$ to GHK, and different from the plane BND.

But two planes passing through the same point D, and $\parallel$ to GHK, must be coincident, that is, the planes BMD, DRF must be the same plane, and this plane must evidently be BDF, for otherwise, two planes not coincident may be made to pass through three points not in the same straight line, which is impossible.

$\therefore$ the plane BDF coincides with both of the planes BMD, DRF, *and is $\parallel$ to GHK* and ACE.

1861. VI. 6. Let the diagonals intersect in O. Join EF, FG, GH HE. In $\triangle AOD$, H and E are two of the feet of perpendiculars, and hence $\triangle$s OEH, OAD are similar,

and $\therefore \angle OHE = \angle ODA$.

Similarly, in the $\triangle$s OAB, OGH,

$\angle OHG = \angle OBA$;

$\therefore \angle EHG = \angle ODA + \angle OBA$,

$= \angle ODA + \angle ODC$,

$= \angle ADC$.

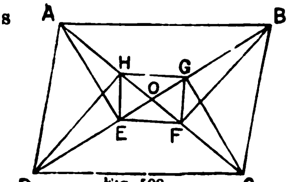


FIG. 598.

Again, since the pairs of $\triangle$s OEH, OAD; OHG OBA, are similar,

$$\therefore EH : AD = OH : DO,$$
$$= OH : BO,$$
$$= HG : BA;$$
$$\text{or, } EH : HG = AD : BA,$$
$$= AD : DC.$$

$\therefore$ the quadrilateral $EFGH$ is similar to $ABCD$, and $\therefore$ is also a parallelogram.

XI. 12. Let $ABCD$ be the tetrahedron, and E the middle pt. of CD; F a pt. in EB such that $FB = 2\,EF$. Join AE, AF.

Let one edge of the tetrahedron be $\sqrt{3}$ in length.

Then, by I. 47, $EB = \frac{3}{2}$, and if G be the middle pt. of AB, $BG = \frac{\sqrt{3}}{2}$.

Now the perpendicular distance between the two lines CD, AB is the shortest distance between them, and this distance is EG.

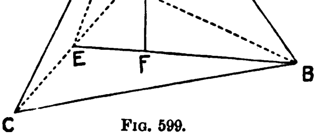


FIG. 599.

Now, by I. 47, sq. on EG + sq. on GB = sq. on EB;

$$\therefore \text{sq. on } EG + \frac{3}{4} = \frac{9}{4}, \text{ and } \therefore EG = \frac{\sqrt{6}}{2}.$$

The diagonal of a square described on an edge = the edge $\times \sqrt{2} = \sqrt{6}$;

$$\therefore \text{half of this diagonal} = \frac{\sqrt{6}}{2} = EG.$$

1862. VI. 1. By a well-known theorem in Modern Geometry, if in the $\triangle$ ABC, AD, BE, CF meet in O.

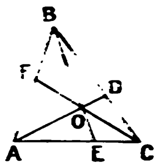


Fig. 600.

$$AE \,.\, BF \,.\, CD = AF \,.\, BD \,.\, CE.$$

Then (1) when $AF : FB = CD : DB$,

we have $AE : CE = AF \,.\, DB : BF \,.\, CD$,

$= CD \,.\, BD : DB \,.\, CD$,

$= 1 : 1$.

And (2) when $AF : FB = BD : DC$,

we have $AE : CE = AF \,.\, BD : BF \,.\, CD$,

$=$ sq. on BD : sq. on CD.

XI. 21. This rider has been already proved in page 335, Ex. 16.

1863. VI. 4. Let ABC be the $\triangle$, AD the external bisector of A, BE and CF the internal bisectors of B and C, meeting the sides in D, E, and F.

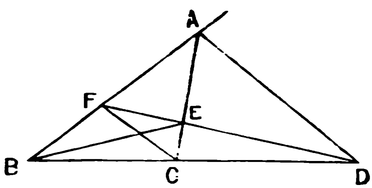


Fig. 601.

Then from the proposition in Modern Geometry, that if three st. lines AD, BE, CF cut the sides of the $\triangle$ ABC in D, E, F, so that DEF is a st. line, then $AE \,.\, BF \,.\, CD = AF \,.\, BD \,.\, CE$;

and since, by VI. 3,

$$CE : EA = CB : BA,$$

$$\text{and } BF : FA = BC : CA,$$

and, by VI. A.,

$$BD : DC = BA : AC;$$

$\therefore$ the equation $AE \,.\, BF \,.\, CD = AF \,.\, BD \,.\, CE$ becomes

$BA \,.\, BC \,.\, AC = CA \,.\, BA \,.\, CB$, an identity,

$\therefore$ DEF is a st. line.

XI. 17. Let $ABCD$ be the tetrahedron, E, F the middle pts. of AD, BC respectively. Join DF, EC, EF, EB.

Then sq. on $DC +$ sq. on $DB = 2$ (sq. on $DF +$ sq. on FC).

By II. 13, Ex.

Also, sq. on DC + sq. on DB = sq. on DE + sq. on EC − 2 rect. $EC.En$
+ sq. on DE + sq. on EB − 2 rect. EB, Em,
= 2 sq. on DE + sq. on EC + sq. on EB − 2 rect. $EC.En$ + 2 rect. EB, Em,
when n, m are the feet of ⊥s from D on EC and EB.

But $DC = AB$, and $DB = AC$.

∴ sq. on DC + sq. on DB = sq. on AB + sq. on AC,
= 2 sq. on EA + sq. on EB + sq. on EC − 2 rect. EB, Em' = 2 rect. EC, En',
where m', n' are the feet of ⊥s from A on BE, CE produced ; also
$Em = Em'$, and $En = En'$.

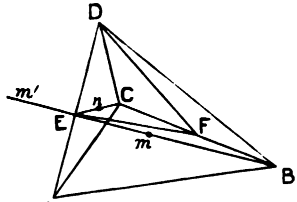


FIG. 602.

∴ adding these two values of (sq. on DC + sq. on DB),
2 (sq. on DC + sq. on DB)
= 2 (sq. on DE + sq. on EA + sq. on EC + sq. on EB),
or, sq. on DC + sq. on DB
= 2 sq. on DE + sq. on EC + sq. on EB (for $EA = DE$) ;
∴ 2 sq. on DF + 2 sq. on FC = 2 sq. on DE + sq. on EC + sq. on EB,
= 2 (sq. on DE + sq. on EF + sq. on FC) ;
∴ sq. on DF = sq. on DE + sq. on EF.

Now DEF is a plane, and ∴, by I. 48, ∠ DEF is a rt. angle.

1864. VI. 23. The ▱s AC, BF are to one another in the ratio compounded of the two ratios $AB : BD$ and $CB : BE$, that is, the ratio $AB \,.\, BC : DB \,.\, BE$; it is required to show that ∠ ABC = ∠ DBE.

If not, let BF' be the ▱ which is to the ▱ AC in the ratio $DB \,.\, BE : AB \,.\, BC$, where ∴ $D'B = DB$ and ∠ $D'BE$ is not = ∠ CBA, that is, ∠ DBE.

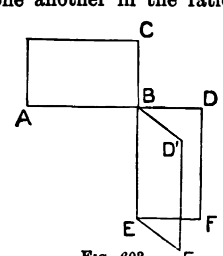


FIG. 603.

And now construct the ▱ DE equiangular with AC, and ∴ such that

$$AC : DE = AB \,.\, BC : DB \,.\, BE.$$

Also, $Ca' : a'B = CA : AB$; (VI. A.)

$$\therefore\ Ca' : CB = CA : AB - CA.$$

Hence $Ca + Ca' = aa' = CA \,.\, CB \,.\left(\frac{1}{CA+AB} + \frac{1}{AB-CA}\right)$;

$$\therefore\ aa' = \frac{2AB \,.\, BC \,.\, CA}{AB^2 - CA^2} = \frac{2AB \,.\, BC \,.\, CA}{BC^2} = 2AB \cdot \frac{AC}{BC}.$$

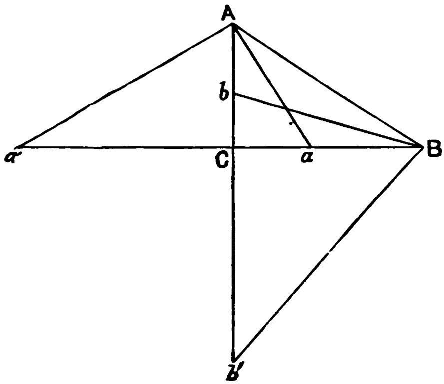


Fig. 608.

Similarly, $bb' = 2 \,.\, AB \cdot \frac{BC}{AC}$;

$$\therefore\ aa' \,.\, bb' = 4AB^2.$$

XI. 21. Looking down from above on the bases of the pyramids, we see two figures like those here shown.

Now $4AB = 8EF$;

$\therefore\ AB = 2EF$.

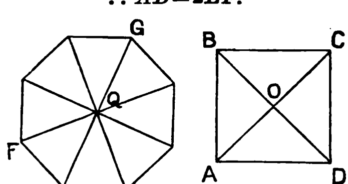


Fig. 609.

Also in the two *sides* (one from each pyramid) AOD, EQF, we have $AO = OD = EQ = QF$.

Bisect AD in H; join OH. Then construct the isosceles $\triangle OAK$, having $OK = OA$ and $AK = AH$. (I. 23.)

Then $\triangle AOK$ will be equal in every respect to $\triangle EQF$.

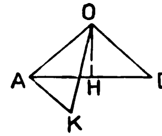


FIG 610.

Now $\angle AOK$ is less than $\angle AOH$;

$\therefore 2 \angle EQF$ is less than $\angle AOD$;

$\therefore$ sum of plane angles at Q is less than sum of plane angles at O.

1868. VI. 2. $\triangle BDE : \triangle BAE = BD : BA$,

$$= BE : BC,$$

$$= \triangle BAE : \triangle BCA;$$

$\therefore \triangle BAE : \triangle BCA = BD : BA$;

$\therefore$, compounding,

$$\triangle BDE : \triangle BCA = BD^2 : BA^2.$$

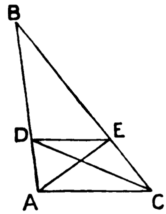


FIG. 611.

1868. XI. 11. This rider consists of two parts—(I.) To construct the "least angle" in question; (II.) To find when this "least angle" is greatest, and what it is.

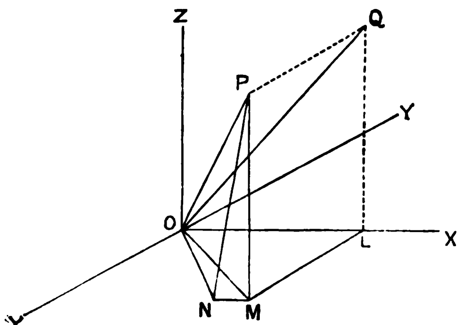


FIG. 612.

(I.) Let OP be the given line in the plane YPY', let YXY' be the other plane, intersecting the former in the line YOY'; it is required to draw from O the line in the plane XOY, which makes the least angle with OP *of all such* lines drawn in the plane XOY.

Draw $OZ \perp$ to XOY; let the plane ZOP cut the plane XOY in OM; from P, any pt. in OP, drop $PM \perp$ to XOY, and $\therefore$ in the plane ZOP, and $\therefore$ meeting OM in some pt. M; $\therefore \angle PMO$ is a right $\angle$. Draw any other line ON in the plane XOY; and draw $PN \perp$ to ON; join MN.

Since PM is $\perp$ to XOY, $\therefore PM$ is $\perp$ to MN, and $\therefore PN$ is greater than PM.

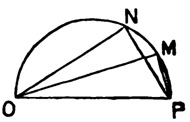


Fig 613.

Now PNO, PMO are two right-angled $\triangle$s having the same hypotenuse OP, and may $\therefore$ be inscribed in the same semicircle of diameter OP, and hence the angle opposite PN in $\triangle OPN$ is at once seen to be greater than the angle opposite PM in $\triangle OPM$;

$\therefore \angle POM$ is less than $\angle PON$.

$\therefore$ the least angle for a given position of OP is that which lies in the plane through $OP \perp$ to the plane XOY.

(II.) Draw PQ, $ML \parallel$ to YO, to meet in Q and L respectively the plane ZOX, which is $\perp$ to YO; join OQ, QL. Then QM is a rectangle, and $QL=PM$. Also PQO is a right angle, and $\therefore PO$ is greater than QO; and MLO is a right angle, and $\therefore MO$ is greater than OL; $\therefore$ in the $\triangle$s QOL, POM, the bases QL and PM are equal, but the sides QO, LO are respectively less than the sides PO, MO; $\therefore$ (as may easily be deduced from I. 20) $\angle POM$ is, for any position of PO passing through O, less than $\angle QOL$ which lies in the plane through $O \perp$ to YOY'. Now OP is in the plane YOP, and PQ is $\parallel$ to YO, $\therefore QO$ is in the plane YOP and $\perp$ to YO; also XO is in the plane XOY, and $\perp$ to YO; $\therefore \angle QOX$ is the angle between the two given planes, and is $\therefore$ seen to be the greatest of the above-mentioned "least angles."

1869. XI. 20. The rider may be proved in three similar steps.

I. Let $A'BC$ be a plane through BC, A' being between A and D.

Then $\angle ABA' + \angle ABC$ is greater than $\angle A'BC$, (XI. 20.)

and $\angle ACA' + \angle ACB$ is greater than $\angle A'CB$, (XI. 20.)

$\therefore \angle ABC + \angle ACB + \angle ABA' + \angle ACA'$ is greater than $\angle A'BC + \angle A'CB$.

Now add $\angle BAD + \angle CAD$ to both sides; then, noting that
$\angle BAD + \angle ABA' = \angle BA'D$, and $\angle CAD + \angle ACA' = \angle CA'D$,
$\angle ABC + \angle ACB + \angle BA'D + \angle CA'D$ is greater than
$\angle A'BC + \angle A'CB + \angle BAD + \angle CAD$.

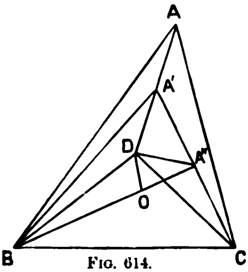


Fig. 614.

Add to the left-hand side $\angle A'BC + \angle A'CB + \angle BA'C = 2$ rt. $\angle$s,
and to the right-hand side $\angle ABC + \angle ACB + \angle BAC = 2$ rt. $\angle$s.
Then cancelling like terms on both sides,
$\angle BA'C + \angle BA'D + \angle CA'D$ is greater than $\angle BAC + \angle BAD + \angle CAD$.

II. Let $A''BD$ be a plane through BD, A'' being between A and C; then, as before,
$\angle BA''C + \angle BA''D + \angle CA''D$ is greater than $\angle BA'C + \angle BA'D + \angle CA'D$,
and $\therefore$ *a fortiori* greater than $\angle BAC + \angle BAD + \angle CAD$.

III. Let OCD be a plane through CD, O being between A'' and B; and therefore, by the construction of I. and II., *within* the tetrahedron. Then we find, as before,
$\angle BOC + \angle BOD + \angle COD$ greater than $\angle BA''C + \angle BA''D + \angle CA''D$,
and $\therefore$ *a fortiori* greater than $\angle BAC + \angle BAD + \angle CAD$.

1870. VI. 15. Let OA, OB be the fixed lines; AFB, $A'FB'$ two lines cutting off equal $\triangle$s AOB, $A'OB'$; E, E' their middle points; CED, XFY, $C'E'D'$ parallels to the given direction.

Then it is required to prove that $CE \cdot ED = C'E' \cdot E'D'$.

By similar $\triangle$s

$$\left.\begin{aligned} CE : XF &= AE : AF \\ ED : FY &= EB : FB \end{aligned}\right\} \text{ and } \left.\begin{aligned} C'E' : XF &= A'E' : A'F \\ E'D' : FY &= E'B' : FB' \end{aligned}\right\};$$

$$\therefore CE \cdot ED : XF \cdot FY = AE \cdot EB : AF \cdot FB;$$

$$\text{and } C'E' \cdot E'D' : XF \cdot FY = A'E' \cdot E'B' : A'F \cdot FB';$$

$$\therefore CE \cdot ED : C'E' \cdot E'D' = AE \cdot EB \times A'F \cdot FB' : A'E' \cdot E'B' \times AF \cdot FB.$$

$$= AE^2 \times A'F \cdot FB' : A'E'^2 \cdot AF \cdot FB.$$

But, by construction, $\triangle AFA' = \triangle BFB'$, and $\therefore$, by VI. 15,
$AF \cdot FA' = BF \cdot FB'$, and $\therefore$ the above proportion becomes
$$CE \cdot ED : C'E' \cdot E'D' = AE^2 \cdot FB'^2 : A'E'^2 \cdot AF^2.$$
Hence the rider is true if
$$AE : A'E' = AF : FB', \text{ for then } CE \cdot ED = C'E' \cdot E'D'.$$

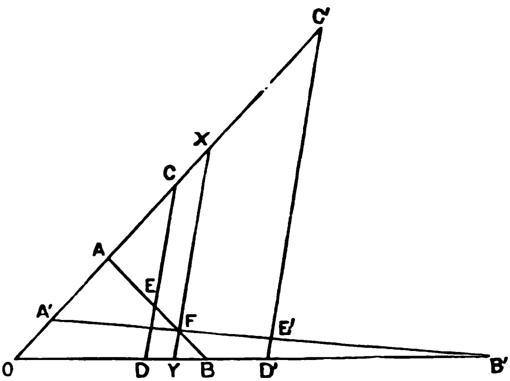


Fig. 615.

Now, if $AE : A'E' = AF : FB'$, then $AE : AF = A'E' : FB'$;
$$\therefore AE : EF = A'E' : E'F \text{ (for } E'B' = A'E');$$
$$\therefore AE - EF : AE + EF = A'E' - E'F : A'E' + E'F,$$
that is, $BF : AF = A'F : FB'$,
that is, $BF \cdot FB' = AF \cdot A'F$,
which is true, as above, by VI. 15;
$$\therefore CE \cdot ED = C'E' \cdot E'D'.$$

XI. 7. Let $ABCD$ be the tetrahedron, such that any two opposite edges are $\perp$ to each other. From A drop a $\perp$ AN on the plane BCD, meeting it in N. Join BN, and produce it to cut CD in B'.

Now $\because$ AN is $\perp$ to the plane BCD, $\therefore$ it is $\perp$ to CD; but, by hypothesis, AB is $\perp$ to CD; and the two lines AB, AN intersect, and $\therefore$ lie in one plane. $\therefore$ the plane in which they lie, viz., ABN, is $\perp$ to CD, and $\therefore$ CD is $\perp$ to BNB' (XI. Def. 3.), that is, BNB', CNC', DND' are the $\perp$s on the sides of $\triangle BCD$; $\therefore$ N is the orthocentre of $\triangle BCD$

Again, if a sphere pass through A, B, C, and D, the plane in which *the* $\triangle BCD$ *lies* will cut the sphere in a circle passing through B, C, D; *let* L *be the* centre of this circle, and Lb, Lc, Ld $\perp$s from L on CD,

DB, BC respectively; b, c, d being $\therefore$ the middle pts. of these sides. Also, by a property of the orthocentre and centre of circumscribing $\odot$ of a triangle, $BN=2Lb$. In the plane ABN bisect BN in M, and draw $MK \parallel$ to NA; then, by VI. 2, K is the middle pt. of BA.

Again, the centre of the circumscribing sphere must lie in the line LO through L and $\perp$ to the plane BCD. Let O be the centre of the circumscribing sphere. Join KO. Let P be the orthocentre of the tetrahedron; join Pb. Then it is required to prove that KO is equal and parallel to Pb.

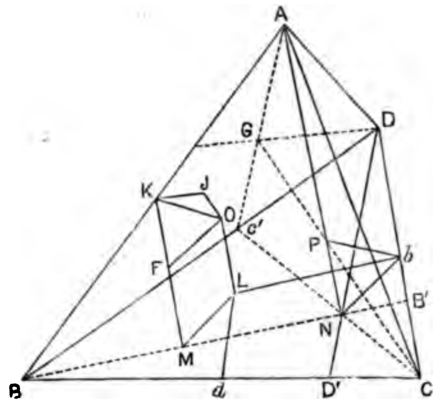


FIG. 616.

First, to prove that KO is $\parallel$ to Pb. The proof will depend on the following almost self-evident proposition :—" If two st. lines lie one in each of two parallel planes, and also one in each of other two parallel planes different from the former two, these two st. lines will be parallel." (See note at end of the proof.) Let G be the orthocentre and J the centre of the $\odot$ circumscribing the $\triangle\, ABD$; then the planes KJO, CGD will be $\parallel$. For KJ is $\perp$ to AB, and JO is $\perp$ to the plane ABD, and $\therefore$ $\perp$ to AB; $\therefore$ AB is $\perp$ to the plane KJO.

And, as before, DG and CG are each $\perp$ to AB, and $\therefore$ AB is $\perp$ to the plane CGD. Hence, by XI. 14, the planes KJO, CGD are parallel to each other, and KO lies in the plane KJO. and Pb lies in the plane CGD.

Again, $\because$ MK, NA are both $\perp$ to the plane BCD, $\therefore$, by XI. 6, they

are ∥. Join ML, Nb, then LN is evidently a ▱, and ∴ ML is ∥ to Nb. Hence, since KM, ML are respectively ∥ to AN, Nb not in the same plane as KM, ML, the plane KML is ∥ to the plane ANb. (XI. 15.) But LO is ⊥ to the plane BCD, as well as MK, and ∴ $KMLO$ is a plane, and is parallel to ANb. Now KO lies in the plane KML, and Pb lies in the plane ANb. Therefore, by the proposition enunciated above, KO and Pb are parallel.

In the plane $KMLO$ draw OF ∥ to ML, meeting KM in F.

Then FL is a ▱; and LN was shown to be one also;

$$\therefore OF = ML = Nb.$$

Now KO is ∥ to Pb, OF is ∥ to Nb, and FK is ∥ to NP.

∴ △s KOF, PbN are similar, and they are also equal, for OF has been proved equal to Nb, and these are homologous sides, and ∴ $KO = Pb$.

Note.—To prove the assumed proposition. Let A, B, C, D be four planes, A ∥ to C, and B ∥ to D; and let A, B intersect in P, and C, D intersect in Q; and B, C intersect in X. It is required to prove that P is ∥ to Q.

By XI. 16, P is ∥ to X; and by the same proposition X is ∥ to Q, ∴ by XI. 9, P is ∥ to Q.

1871. VI. 2. By two well-known theorems of Modern Geometry,

(1.) Since AD, BE, CF meet in O,

$$AE \cdot CD \cdot BF = AF \cdot BD \cdot CE;$$

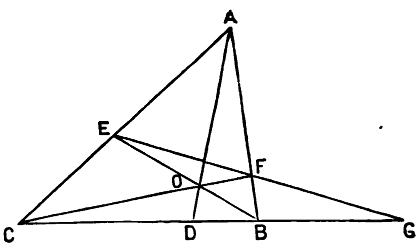


FIG. 617.

(2.) Since E, F, G are in the same line,

$$AE \cdot CG \cdot BF = AF \cdot BG \cdot CE;$$

$$\therefore AE \cdot CD \cdot BF \times AF \cdot BG \cdot CE = AF \cdot BD \cdot CE \times AE \cdot CG \cdot BF;$$

$$\therefore CD \cdot BG = BD \cdot CG;$$

$$\therefore BD : DC = BG : GC.$$

XI. 11. Let $ABCD$ be the tetrahedron; DN, CM two of the $\perp$s meeting in O. CM and DN will not meet unless the planes CDM, DCN coincide, that is, unless DM when produced meets CN when produced in one and the same point L in AB. Now the coincidence of these two planes, and the perpendicularity of AB to the coincident planes (for DN is $\perp$ to AB, and CM is also $\perp$ to AB), are co-extensive conditions, and $\therefore$ the $\perp$s CM, DN will not meet unless AB be $\perp$ to their common plane CLD, that is, unless $DB^2 - DA^2 = LB^2 - LA^2$. This condition

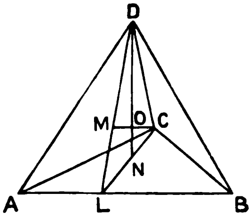


FIG. 618.

might also be expressed under the form unless $CB^2 - CA^2 = LB^2 - LA^2$. Either of these conditions is necessary, and if the one holds the other must hold also.

However, the pt. L is undetermined; we may then use these two conditions to get rid of reference to the pt. L; doing this, the conditions become the single condition $CB^2 - CA^2 = DB^2 - DA^2$, that is, $BC^2 + DA^2 = AC^2 + BD^2$. Hence the given condition is *necessary*.

It is also *sufficient*. For suppose a point L constructed for the $\perp$ DN, and a point L' for the $\perp$ CM; then from the condition would

$$CB^2 - CA^2 = DB^2 - DA^2,$$

while from these constructions would

$$CB^2 - CA^2 = L'B^2 - L'A^2, \text{ and } DB^2 - DA^2 = LB^2 - LA^2,$$

$$\therefore L'B^2 - L'A^2 = LB^2 - LA^2.$$

Now, by the axiom that the whole = sum of the parts,

$$\therefore L'B + L'A = LB + LA;$$

$$\therefore L'B - L'A = LB - LA;$$

$$\therefore 2\,L'B = 2\,LB;$$

$\therefore$ L and L' are the same point, $\therefore$ the plane $CDNL$ coincides with the plane $DCML'$, and $\therefore$ the $\perp$s CM, DN intersect.

COR. I. It may be noticed that when two $\perp$s CM, DN intersect, their plane is $\perp$ to the line of intersection of the two faces to which each is $\perp$; but CD is in this plane, and AB is the line of intersection of these two faces. Hence, when the $\perp$s intersect, the opposite edges are perpendicular.

COR. II. When the $\perp$s intersect, N is the orthocentre of the $\triangle\,ABC$, M that *of the* $\triangle\,ABD$.

1572. VI. 2. Let PCP' be the ⊙, NQ the st. line, O the pt.: $m : n$ the given ratio.

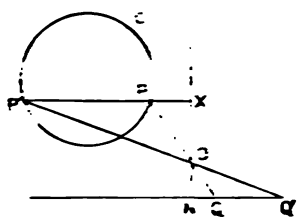


Fig. 619.

Draw $ON \perp$ to NQ, cutting it in N, and produce NO to X, so that

$$ON : OX = m : n.$$

Through X draw $XPP' \parallel$ to NQ, to cut the ⊙ in P and P'.

Draw POQ, $P'OQ'$ to cut the given st. line in Q and Q'; then POQ, $P'OQ'$ are each cut by O in the ratio $n : m$.

For XP is $\parallel$ to NQ,

and $\therefore PO : OQ = XO : ON = n : m$.

Hence the construction is proved when O divides PQ *internally* in the given ratio.

The annexed figure shows O dividing QP *externally* in the given ratio. In each case there are seen to be two solutions except in the

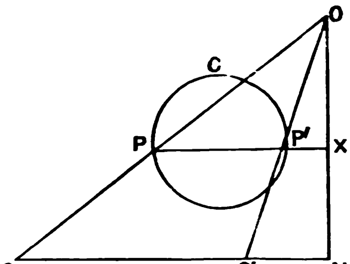


Fig. 620.

limiting positions, when the two coincide, *i.e.* when XPP' is a tangent to the ⊙.

The next figure shows the possibility of four solutions, two internal and two external, when O lies between the two tangents to the ⊙, ∥ to NQ.

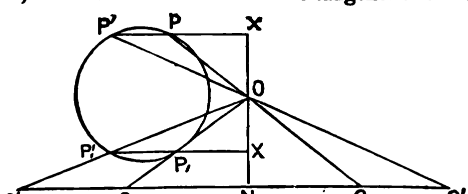


Fig. 621.

The same reasoning holds for the cases in which O lies *within* the ⊙.

It may be noticed that for a given position of X and a given value of $m:n$, two positions of O may be found. Thus, in this fourth figure, take $ON:OX=m:n$ and two lines are found, of which POQ is one. Again, take $O'X:NO'=m:n$, and other two lines may be found, of which $PO'Q'$ is one.

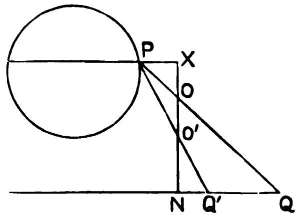


FIG. 622.

For the limits of the ratio $m:n$ there will be three cases to consider.

Draw CA, DB tangents ∥ to the given line, cutting NO in A and B respectively.

CASE I. When O lies between N and B, X must lie between B and A, and ∴ if we take $NO:OX=m:n$, we must have OX equal to or greater than OB, and less than or equal to OA, ∴ $NO:OX$ is

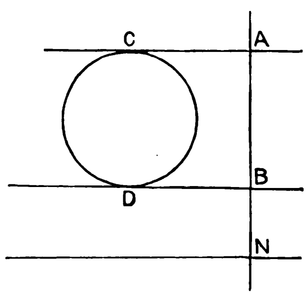


FIG. 623.

equal to or less than $NO:OB$, and greater than or equal to $NO:OA$, that is, $m:n$ must not exceed the limits $NO:OA$ and $NO:OB$.

CASE II. When O lies between B and A, then as before X is also between B and A, and OX must be equal to or less than OB, $\left.\begin{matrix}\text{or}\\\text{and}\end{matrix}\right\}$ equal to or less than OA; that is, $m:n=NO:OX$ is equal to or greater than $NO:OB$, $\left.\begin{matrix}\text{or}\\\text{and}\end{matrix}\right\}$ equal to or greater than $NO:OA$;

that is, $m : n$ must not be less for external section than $NO : OB$, and not less for internal section than $NO : OA$.

CASE III. When O lies further from N than A, then OX must be equal to or greater than OA, and less than or equal to OB; that is, $m : n = NO : OX$ must be equal to or less than $NO : OA$, and greater than or equal to $NO : OB$; that is, $m : n$ must not lie beyond the limits $NO : OA$ and $NO : OB$.

The point O might also lie on the other side of NQ from that on which the circle lies. This case is included in the former cases; and the limits of the ratio $m : n = NO : OX$ are $NO : OA$ and $NO : OB$.

XI. 20. PART I. $ABCD$ is the tetrahedron; $AB = CD$; $BD = CA$; and AD is common to $\triangle$s ABD, DCA;

$$\therefore \angle ABD = \angle ACD. \qquad (1.)$$

Similarly, $\angle ABC = \angle ADC$, and $\angle DBC = \angle DAC$. (2.)

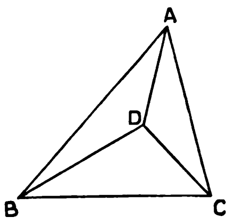


FIG. 624.

Now by XI. 20, any two of the $\angle$s at B are together greater than the third. Let $\angle$s ABD, DBC be together greater than $\angle CBA$, that is, by what has been proved, $\angle ACD + \angle DAC$ is greater than $\angle ADC$, *i.e.* the three $\angle$s ACD, CDA, and DAC are together greater than $2 \angle CDA$;

or, $2 \angle CDA$ is less than 2 rt. $\angle$s;

$\therefore \angle CDA$ is less that a rt. $\angle$.

The same holds for all the other plane $\angle$s.

Therefore each of the four triangles ABC, ACD, ADB, BCD is acute-angled.

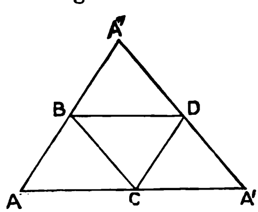


FIG. 625.

PART II. Let the four equal and similar acute-angled $\triangle$s be laid in one plane, as in the diagram, thus (the order of the letters indicating the equal sides), BDC, $A'CD$, DBA'', $CA'''B$.

A little consideration will show that this is the only possible relative position of the $\triangle$s. For evidently the side BD of BDA'' must lie along and coincide with BD *of the* $\triangle$ BDC. Suppose BD of BDA'' were reversed, then the *side* BA'' (on an attempt to form the tetrahedron) would have to

coincide with DA', to which it is not equal. Therefore this is the only relative position. The only condition to which the three plane angles containing a solid angle are subject is (XI. 20) that *any* two are together greater than the third. Now if BDC be *not* acute-angled, let BDC be the angle greater than a right $\angle$; then $\angle A''DB$ is greater than a right $\angle$, and the $\angle$s BDC, DCB, *i.e.* $\angle$s BDC and CDA', are together less than a right $\angle$, and therefore less than $\angle A''DB$. Hence three such $\angle$s cannot enclose a solid $\angle$, and $\therefore$ the four equal and similar $\triangle$s must be acute-angled.

When this is the case, let the $\triangle$s $A''BD$, BDC, CDA' be so placed that DA'' coincides with DA', and since $DA''=DA'$, $\therefore$ A'' will coincide with A'; call this point A. Then we have BA, AC in one plane, and $\therefore$ BC in the same plane; and $\therefore$ BCA is a plane $\triangle$, and $BA=BA'''$, $CA=CA'''$, and BC is common to the two $\triangle$s BCA, BCA''', and so $\triangle BCA'''$ may be made to have its sides respectively coinciding with the sides of $\triangle BCA$, that is, BA with BA''', and CA with CA''', and as these sides are equal, A''' will coincide with A. Hence four equal and similar acute-angled triangles can be made into a tetrahedron.

Edinburgh University Press:
T. AND A. CONSTABLE, PRINTERS TO HER MAJESTY.

By the same Author

Crown 8vo. 3*s.* 6*d.*

An Elementary Latin Grammar.

Crown 8vo. 3*s.* 6*d.*

Exercises on the Elementary Principles of Latin Prose Composition. With Examination Papers on the Elementary Facts of Latin Accidence and Syntax.

A Key. Crown 8vo. 5*s.*

Crown 8vo. 4*s.* 6*d.*

An Elementary Greek Grammar.

Crown 8vo. 2*s.* 6*d.*

Notes on the Greek Text of the Gospel of St. Mark.

Crown 8vo. 4*s.* 6*d.*

Notes on the Greek Text of the Acts of the Apostles.

Small 8vo. 2*s.* 6*d.*

The Rudiments of English Grammar and Composition.

3, WATERLOO PLACE, LONDON,
June, 1884.

EDUCATIONAL WORKS

PUBLISHED BY

MESSRS. RIVINGTON

NEW BOOKS IN PREPARATION AND IN THE PRESS

Letters of Cicero. Selected and Edited, with Introduction and Notes. *By* J. H. MUIRHEAD, B.A., Oxon., *Assistant to Professor of Humanity in the University of Glasgow. Crown* 8*vo.* [*In the press.*

Cicero Pro Cluentio. With Introduction, Notes, &c. *By* W. YORKE FAUSSETT, B.A., *Assistant Master in the Grammar School, Manchester.* [*In preparation.*

Latin Prose Exercises. Consisting of about 250 easy Passages for Translation. *By* T. L. PAPILLON, M.A., *Fellow and Tutor of New College, Oxford.* [*In preparation.*

An Elementary Greek Syntax. *By* F. E. THOMPSON, M.A., *Assistant Master at Marlborough College; Author of* "A Syntax of Attic Greek for the use of Students," &c. *Crown* 8*vo.* [*In the press.*

Selections from Thucydides. For the use of Middle Forms of Schools. *Edited by* E. H. MOORE, M.A., *Assistant Master in the High School, Plymouth.* [*In preparation.*

Passages for Unseen French Translation. *By* C. H. PARRY, M.A., *Assistant Master at Charterhouse. Crown* 8*vo.* [*In the press.*

Exercises in French Syntax. With Rules. *By* G. SHARP, M.A., *Assistant Master at Marlborough College. Crown* 8*vo.* [*In the press.*

Latin Passages adapted for Practice in Unseen Translation. For the use of Middle and Upper Forms of Schools. *By* J. Y. SARGENT, M.A., *Fellow and Tutor of Hertford College, Oxford; and Editor of* "Materials and Models for Latin and Greek Prose Composition." *Crown* 8*vo.* 2*s.* 6*d.* [*Now ready.*

New Books in Preparation and in the Press—*continued.*

Passages for Latin and Greek Unseen Translation. For the use of Lower and Middle Forms of Schools. *By* J. ARNOLD TURNER, B.A., *Assistant Master at Hillbrow School, Rugby. Small 8vo.* [*In preparation.*

A Collection of Arithmetical Exercises, progressively arranged. *By* A. E. DONKIN, M.A., *and* C. H. HODGES, M.A., *Assistant Masters at Rugby School. Small 8vo.* [*In the press.*

A History of the Romans. For the use of Middle Forms of Schools. *By* R. F. HORTON, M.A., *Fellow and Lecturer of New College, Oxford. Crown 8vo.* [*Just ready.*

A History of Hellas, from the Earliest Times to the Death of Alexander. For the use of Upper Forms of Schools. *By* EVELYN ABBOTT, M.A., LL.D., *Fellow and Tutor of Balliol College, Oxford. Crown 8vo.* [*In preparation.*

History of the Romans to the Establishment of Imperialism. *By* J. S. REID, LL.M., *Fellow and Assistant Tutor of Gonville and Caius College, Cambridge; Classical Examiner in the University of London.* [*In preparation.*

A History of England for Schools. With Maps and Illustrations. *By* F. YORK-POWELL, M.A., *Lecturer at Christ Church, Oxford, and* J. M. MACKAY, M.A., *Master of the English Department in the High School, Edinburgh. Crown 8vo.* [*In the press.*

A History of England. For the use of Middle Forms of Schools. With Contents, Tables, Plans, Maps, Index, &c. *By* CYRIL RANSOME, M.A., *Professor of Modern Literature and History, Yorkshire College, Leeds. Small 8vo.* [*In preparation.*

Viri Illustres Urbis Romae. An Elementary Latin Reading Book. With Notes and Vocabulary. *By* G. L. BENNETT, M.A., *Head Master of Sutton Valence School. Small 8vo.* [*In the press.*

A Preparatory Latin Reader. With Notes and a Vocabulary. *By* F. RITCHIE, M.A., *The Beacon, Sevenoaks, and late Assistant Master in the High School, Plymouth. Crown 8vo.* [*In the press.*

A German Exercise Book. Containing about 150 Exercises, with the necessary Accidence and Syntax, and a Vocabulary. *By* W. G. GUILLEMARD, M.A., *Assistant Master at Harrow School.*

Scott's Marmion. With Introduction, Notes, &c. *By* F. S. ARNOLD, M.A., *Assistant Master at Bedford Grammar School.* Forming a Volume of "English School Classics." *Small 8vo.*

New Books in Preparation and in the Press—*continued.*

Builders' Work and the Building Trades. *By* Lieut.-Colonel H. C. SEDDON, R.E., *Examiner in Building Construction, Science and Art Department, South Kensington; Assistant Examiner, H.M. Civil Service Commissioners; Instructor in Construction, School of Military Engineering, Chatham.* [*In the press.*

This volume contains descriptions of work performed by the different Building Trades; the ordinary Tools and Materials used; methods of executing, measuring, and valuing Builders' work; together with useful data and practical hints connected with the same.

Highways of History. A Series of Volumes on portions of English History by various writers. *Edited by* LOUISE CREIGHTON, *Author of* "A First History of England," "Stories from English History," &c. *Small 8vo.* [*In the press.*

THE GOVERNMENT OF ENGLAND. 1*s.* 6*d.*

THE CONNECTION BETWEEN ENGLAND AND SCOTLAND.

THE HISTORY OF RELIGION IN ENGLAND.

THE RELATIONS OF ENGLAND WITH FOREIGN POWERS.

ENGLAND AND IRELAND. 1*s.* 6*d.*

THE SOCIAL HISTORY OF ENGLAND.

THE GROWTH OF THE ENGLISH COLONIES. 1*s.* 6*d.*

The aim of this series is to give consecutive treatment to certain important subjects in English History. A strictly chronological arrangement of history distracts the reader's attention from one subject to another. Though England's progress has to be studied by reading her history as a whole, yet clearness is gained on many points by a separate survey of some important line of advance. The treatment of the subjects in this series will be adapted for the use of students who have already mastered the general outline of English History, but wish for more connected information with regard to some special point. Each part will be complete in itself, and will treat of its own subject without reference, except where necessary, to the general course of events.

Greek Passages adapted for practice in Unseen Translation. Intended for the use of Middle and Higher Forms of Schools, and for University and other Students. *By* ARTHUR SIDGWICK, M.A., *Fellow of Corpus Christi College, Oxford; late Assistant Master at Rugby School, and Fellow of Trinity College, Cambridge.* [*In preparation.*

A Latin-English Dictionary for Junior Forms of Schools. *By* C. G. GEPP, M.A., *Assistant Master at Bradfield College.* 16*mo.* [*In the press.*

Bacon's Essays. Complete Edition. *By* FRANCIS STORR, P
Forming a Volume of "English School Classics." *Small* 8
[*Ir*

KEYS

KEYS are published to the following Educational Works *for the use of Tutors only.* They can *only* be obtained by direct application to the Publishers, who will send a printed Form, to be filled up by the Tutor requiring the KEY. They cannot be supplied through Booksellers.

		Nett Price of the Key. *s.*	*d.*
ABBOTT'S	Arnold's Greek Prose Composition	3	6
AINGER'S	Clivus. Parts I. and II.	3	6
ARNOLD'S	Henry's First Latin Book	1	0
———	Second Latin Book	2	0
———	First Verse Book	1	0
———	Latin Prose Composition. Part I.	1	6
———	First Greek Book	1	6
———	Greek Prose Composition. Part I.	1	6
BENNETT'S	First Latin Writer	5	0
———	Second Latin Writer	5	0
———	Easy Latin Stories for Beginners	5	0
———	Second Latin Reading Book	5	0
———	Passages for Latin Unseen Translation	3	6
BRADLEY'S	Aids to Writing Latin Prose	5	0
———	Arnold's Latin Prose Composition	5	0
GEPP'S	Arnold's Henry's First Latin Book	5	0
———	Exercises in Latin Elegiac Verse	5	0
HEATLEY AND KINGDON'S	Gradatim	5	0
———	Excerpta Facilia	5	0
HEATLEY'S	Graecula	5	0
MORICE'S	Arnold's First Greek Book	5	0
RAVEN'S	Latin Grammar Papers	5	0
RITCHIE AND MOORE'S	Greek Method	5	0
RITCHIE'S	First Steps in Latin	3	6
SARGENT AND DALLIN'S	Materials and Models for Latin Prose Composition. Latin Version. 116 Selected Pieces	5	0
———	Greek Prose Composition. Greek Version. 92 Pieces	7	6
SIDGWICK'S	First Greek Writer	5	0
———	Greek Prose Composition	5	0
SIDGWICK AND MORICE'S	Greek Verse Composition	5	0

Keys to the following are sold to the Public without restriction.

ARNOLD'S	First German Book	2	6
———	First French Book	2	6
———	First Italian Book	1	6
———	First Hebrew Book	3	6
SMITH'S (J. HAMBLIN)	Elementary Algebra	9	0
———	Arithmetic	9	0
———	Geometry	8	6
———	Statics and Hydrostatics	6	0
———	Trigonometry	7	6
———	Latin Prose Composition	5	0
SPRATT AND PRETOR'S	Latin and Greek Translation at Sight	0	0

ENGLISH

Select Plays of Shakspere. RUGBY EDITION.

With Introduction and Notes to each Play. Small 8vo.

As You Like It. *2s.* King Lear. *2s. 6d.*

Hamlet. *2s. 6d.* Macbeth. *2s.*

Romeo and Juliet. *2s.* King Henry the Fifth. *2s.*

A Midsummer Night's Dream. *2s.* King John. *2s.*

Edited by C. E. MOBERLY, M.A., *formerly Scholar of Balliol College, Oxford.*

Coriolanus. *2s. 6d. Edited by* ROBERT WHITELAW, M.A., *Assistant Master at Rugby School.*

The Tempest. *2s. Edited by* J. SURTEES PHILLPOTTS, M.A., *Head Master of Bedford Grammar School.*

Small 8vo. 2s.

Shakspere's Othello.

With Introduction and Notes. *Edited by* E. K. PURNELL, M.A., *Assistant Master at Wellington College.*

Highways of History.

A Series of Volumes on portions of English History, by various writers. *Edited by* LOUISE CREIGHTON, *Author of* "A First History of England," &c. *Small 8vo.*

The Government of England. *1s. 6d.*

The Connection between England and Scotland.

The History of Religion in England.

The Relations of England with Foreign Powers.

England and Ireland. *1s. 6d.*

The Social History of England.

The Growth of the English Colonies. *1s. 6d.*

English School Classics

With Introductions, and Notes at the end of each Book.

Edited by FRANCIS STORR, B.A.,

CHIEF MASTER OF MODERN SUBJECTS AT MERCHANT TAYLORS' SCHOOL.

Small 8vo.

Thomson's Seasons: Winter. *With Introduction to the Series, by the* REV. J. FRANCK BRIGHT, D.D., *Master of University College, Oxford.* 1*s.*

Cowper's Task. *By* FRANCIS STORR, B.A. 2*s.*
Books I. and II., 9*d.*; Books III. and IV., 9*d.*; Books V. and VI., 9*d.*

Cowper's Simple Poems. *With Life of the Author. By* FRANCIS STORR, B.A. 1*s.*

Scott's Lay of the Last Minstrel. *By* J. SURTEES PHILLPOTTS, M.A., *Head Master of Bedford School.* 2*s.* 6*d.*
Canto I., 9*d.*; Cantos II. and III., 9*d.*; Cantos IV. and V , 9*d.*; Canto VI., 9*d.*

Scott's Lady of the Lake. *By* R. W. TAYLOR, M.A., *Head Master of Kelly College, Tavistock.* 2*s.*
Cantos I. and II., 9*d.*; Cantos III. and IV., 9*d.*; Cantos V. and VI., 9*d.*

Scott's Marmion. *By* F. S. ARNOLD, M.A., *Assistant Master at Bedford School.* [*In preparation.*

Notes to Scott's Waverley. *By* H. W. EVE, M.A., *Head Master of University College School, London.* 1*s.*; *or with the Text,* 2*s.* 6*d.*

Bacon's Essays. Complete Edition. *By* FRANCIS STORR, B.A. [*In the press.*

Twenty of Bacon's Essays. *By* FRANCIS STORR, B.A. 1*s.*

Simple Poems. *Edited by* W. E. MULLINS, M.A., *Assistant Master at Marlborough College.* 8*d.*

Selections from Wordsworth's Poems. *By* H. H. TURNER, B.A., *late Scholar of Trinity College, Cambridge.* 1*s.*

Wordsworth's Excursion: The Wanderer. *By* H. H. TURNER, B.A. 1*s.*

Milton's Paradise Lost. *By* FRANCIS STORR, B.A. Book I., 9*d.* Book II., 9*d.*

Milton's L'Allegro, Il Penseroso, and Lycidas. *By* EDWARD STORR, M.A., *late Scholar of New College, Oxford.* 1*s.*

Selections from the Spectator. *By* OSMOND AIRY, M.A., *late Assistant Master at Wellington College.* 1*s.*

Browne's Religio Medici. *By* W. P. SMITH, M.A., *Assistant Master at Winchester College.* 1*s.*

Goldsmith's Traveller and Deserted Village. *By* C. SANKEY, M.A., *Head Master of Bury St. Edmund's Grammar School.* 1*s.*

Extracts from Goldsmith's Vicar of Wakefield. *By* C. SANKEY, M.A. 1*s.*

Poems selected from the Works of Robert Burns. *By* A. M. BELL, M.A., *Balliol College, Oxford.* 2*s.*

Macaulay's Essays.

MOORE'S LIFE OF BYRON. *By* FRANCIS STORR, B.A. 9*d.*

BOSWELL'S LIFE OF JOHNSON. *By* FRANCIS STORR, B.A. 9*d.*

HALLAM'S CONSTITUTIONAL HISTORY. *By* H. F. BOYD, *late Scholar of Brasenose College, Oxford.* 1*s.*

Southey's Life of Nelson. *By* W. E. MULLINS, M.A. 2*s.* 6*d.*

Gray's Poems. Selection from Letters, with Life by Johnson. *By* FRANCIS STORR, B.A. 1*s.*

New Edition. Small 8vo. 2*s.* 6*d.*

The Rudiments of English Grammar and Composition. *By* J. HAMBLIN SMITH, M.A., *of Gonville and Caius College, and late Lecturer at St. Peter's College, Cambridge.*

Small 8vo. 1*s.* 6*d.*

A Primer of English Parsing and Analysis. *By* CYRIL L. C. LOCKE, M.A., *Assistant Master at Clifton College.*

Small 8vo. 1*s.* 6*d.*

The Beginner's Drill-book of English Grammar. Adapted for Middle Class and Elementary Schools. *By* JAMES BURTON, B.A., *First English Master in the High School of the Liverpool Institute.*

Small 8vo. 2*s.* 6*d.*

A Practical English Grammar. For Schools and Colleges, and for Students preparing for examinations. *By the* Rev. W. TIDMARSH, B.A., *late Head Master of Putney School.*

Small 8vo. 2*s.* 6*d.*

Short Readings in English Poetry. Arranged, with occasional Notes, for the use of Schools and Classes. *Edited by* H. A. HERTZ.

HISTORY

With Maps and Plans. New Edition, Revised. Crown 8vo.

A History of England. *By the* Rev. J. FRANCK BRIGHT, D.D., *Master of University College, Oxford, and late Master of the Modern School at Marlborough College.*

PERIOD I.—MEDIÆVAL MONARCHY: The departure of the Romans, to Richard III. From A.D. 449 to A.D. 1485. 4s. 6d.

PERIOD II.—PERSONAL MONARCHY: Henry VII. to James II. From A.D. 1485 to A.D. 1688. 5s.

PERIOD III.—CONSTITUTIONAL MONARCHY: William and Mary, to the present time. From A.D. 1689 to A.D. 1837. 7s. 6d.

Crown 8vo. 6s.

The Rise of Constitutional Government in England. *By* CYRIL RANSOME, M.A., *Professor of Modern Literature and History, Yorkshire College, Leeds.*

With Maps and Illustrations. Crown 8vo.

A History of England for Schools. *By* F. YORK-POWELL, M.A., *Lecturer at Christ Church, Oxford; and* J. M. MACKAY, M.A., *Master of the English Department in the High School, Edinburgh.* [*In the press*

Second Edition. With Forty Illustrations. 16mo. 2s. 6d.

A First History of England. *By* LOUISE CREIGHTON, *Author of* "Life of the Black Prince," "Sir Walter Ralegh," &c.

With numerous Illustrations. Royal 16mo. 3s. 6d.

Stories from English History. *By* LOUISE CREIGHTON, *Author of* "A First History of England," "Life of the Black Prince," &c.

New Edition. 18mo. 1s. 6d.

A History of England for Children. *By* GEORGE DAVYS, D.D., *formerly Bishop of Peterborough.*

Historical Handbooks. *Edited by* OSCAR BROWNING, M.A., *Fellow of King's College, Cambridge.*

Crown 8vo.

English History in the XIVth Century. *By* CHARLES H. PEARSON, M.A., *late Fellow of Oriel College, Oxford.* 3*s*. 6*d*.

The Reign of Lewis XI. *By* P. F. WILLERT, M.A., *Fellow of Exeter College, Oxford.* With Map. 3*s*. 6*d*.

The Roman Empire. A.D. 395–800. *By* A. M. CURTEIS, M.A. With Maps. 3*s*. 6*d*.

History of the English Institutions. *By* PHILIP V. SMITH, M.A., *Fellow of King's College, Cambridge.* 3*s*. 6*d*.

History of Modern English Law. *By* Sir R. K. WILSON, Bart., M.A., *late Fellow of King's College, Cambridge.* 3*s*. 6*d*.

History of French Literature. *Adapted from the French of* M. DEMOGEOT, *by* C. BRIDGE. 3*s*. 6*d*.

Crown 8vo.

A History of the Romans. For the use of Middle Forms of Schools. *By* R. F. HORTON, M.A., *Fellow and Lecturer of New College, Oxford.* [*In the press.*

Crown 8vo. 7s. 6d.

Chapters in the History of English Literature. *From 1509 to the close of the Elizabethan Period.*

By ELLEN CROFTS, *Lecturer at Newnham College, Cambridge.*

Crown 8vo. 7s. 6d.

Ecclesia Anglicana. A History of the Church of Christ in England from the earliest to the present times. *By* ARTHUR CHARLES JENNINGS, M.A., *Jesus College, Cambridge; Vicar of Whittlesford.*

Second Edition, Revised. Crown 8vo. 7s. 6d.

History of the Church under the Roman Empire, A.D. 30–476. *By the* Rev. A. D. CRAKE, B.A., *Vicar of Haven Street, Ryde.*

Historical Biographies. *Edited by the* Rev. M. CREIGHTON, M.A., LL.D., *late Fellow and Tutor of Merton College, Oxford.* With Maps and Plans. *Small 8vo.*

Simon de Montfort. *By* M. CREIGHTON, M.A., LL.D. 2*s*. 6*d*.
The Black Prince. *By* LOUISE CREIGHTON. 2*s*. 6*d*.
Sir Walter Ralegh. *By* LOUISE CREIGHTON. 3*s*.
Oliver Cromwell. *By* F. W. CORNISH, M.A. 3*s*. 6*d*.
The Duke of Marlborough. *By* LOUISE CREIGHTON. 3*s*. 6*d*.
The Duke of Wellington. *By* ROSAMOND WAITE. 3*s*. 6*d*.

Second Edition. Crown 8vo. 6*s*.

A Handbook in Outline of the Political History of England to 1882. Chronologically arranged. *By* A. H. DYKE ACLAND, M.A., *Christ Church, Oxford*, and CYRIL RANSOME, M.A., *Professor of Modern Literature and History, Yorkshire College, Leeds.*

Crown 8vo. Paper cover, 1*s*. *Cloth limp,* 1*s*. 6*d*.

A Handbook in Outline of English Politics for the Last Half Century. Extracted from "A Handbook of English Political History." With Appendices on the Reform Bills, Disfranchised and Enfranchised Boroughs, &c. *By* A. H. DYKE ACLAND, M.A., *and* CYRIL RANSOME, M.A.

Small 8vo. 1*s*. 6*d*.

A Skeleton Outline of the History of England, being an abridgment of a Handbook in Outline of the Political History of England. *By* A. H. DYKE ACLAND, M.A., *and* CYRIL RANSOME, M.A.

Small 8vo. 2*s*. 6*d*.

A Skeleton Outline of Greek History. Chronologically arranged. *By* EVELYN ABBOTT, M.A., LL.D., *Fellow and Tutor of Balliol College, Oxford.*

Small 8vo. 2*s*.

A Skeleton Outline of Roman History. Chronologically arranged. *By* P. E. MATHESON, M.A., *Fellow of New College, Oxford.*

SCIENCE

Third and Enlarged Edition. With Illustrations. 8vo. 21s.

Physical Geology for Students and General Readers.

By A. H. GREEN, M.A., F.G.S., *Professor of Geology in the Yorkshire College of Science, Leeds.*

Crown 8vo. 2s. 6d.

Elementary Course of Practical Physics.

By A. M. WORTHINGTON, M.A., F.R.A.S., *Assistant Master at Clifton College.*

New Edition, Revised. With Illustrations. Crown 8vo. 2s. 6d.

An Easy Introduction to Chemistry.

Edited by the Rev. ARTHUR RIGG, M.A., *and* WALTER T. GOOLDEN, M.A., *Lecturer in Natural Science at Tonbridge School.*

Second Edition. With Illustrations. Crown 8vo. 5s.

A Year's Botany.

Adapted to Home and School Use. *By* FRANCES ANNE KITCHENER. Illustrated by the Author.

With Illustrations. Medium 8vo.

Notes on Building Construction.

Arranged to meet the requirements of the syllabus of the Science and Art Department of the Committee of Council on Education, South Kensington.

PART I.—FIRST STAGE, OR ELEMENTARY COURSE. Second Edition. *With* 325 *woodcuts.* 10s. 6*d.*

PART II.—COMMENCEMENT OF SECOND STAGE, OR ADVANCED COURSE. Second Edition. *With* 300 *woodcuts.* 10*s.* 6*d.*

PART III.—ADVANCED COURSE. *With* 188 *woodcuts.* 21*s.*

REPORT ON THE EXAMINATION IN BUILDING CONSTRUCTION, HELD BY THE SCIENCE AND ART DEPARTMENT, SOUTH KENSINGTON, IN MAY, 1875.—"*The want of a text-book on this subject, arranged in accordance with the published syllabus, and therefore limiting the students and teachers to the prescribed course, has lately been well met by a work published by Messrs. Rivingtons, entitled* 'Notes on Building Construction,' *arranged to meet the requirements of the Syllabus of the Science and Art Department of the Committee of Council on Education, South Kensington.*"

(*Signed*) H. C. SEDDON, Major R.E.

June 18, 1875. [*Instructor in Construction and Estimating at the School of Military Engineering, Chatham.*]

MATHEMATICS

Rivington's Mathematical Series.

Small 8vo. 3*s.* *Without Answers*, 2*s.* 6*d.*

Elementary Algebra. *By* J. HAMBLIN SMITH, M.A., *of Gonville and Caius College, and late Lecturer in Classics at St. Peter's College, Cambridge.*

Also an Edition especially prepared to cover the ground required by the Regulations for the University Examinations in India. 3*s.*

A KEY TO ELEMENTARY ALGEBRA. 9*s.*

Small 8vo. 2*s.* 6*d.*

Exercises on Algebra. *By* J. HAMBLIN SMITH, M.A.
(Copies may be had without the Answers.)

Crown 8vo. 8*s.* 6*d.*

Algebra. PART II. *By* E. J. GROSS, M.A., *Fellow of Gonville and Caius College, Cambridge, and Secretary to the Oxford and Cambridge Schools Examination Board.*

Small 8vo. 3*s.* 6*d.*

A Treatise on Arithmetic. *By* J. HAMBLIN SMITH, M.A.
(Copies may be had without the Answers.)

A KEY TO ARITHMETIC. 9*s.*

Small 8vo. 4*s.* 6*d.*

Elementary Trigonometry. *By* J. HAMBLIN SMITH, M.A.

A KEY TO ELEMENTARY TRIGONOMETRY. 7*s.* 6*d.*

Crown 8vo. 5*s.* 6*d.*

Kinematics and Kinetics. *By* E. J. GROSS, M.A.

Crown 8vo. 4*s.* 6*d.*

Geometrical Conic Sections. *By* G. RICHARDSON, M.A., *Assistant Master at Winchester College.*

Small 8vo. 3*s.*

Elementary Statics. *By* J. HAMBLIN SMITH, M.A.

Small 8vo. 3*s.*

Elementary Hydrostatics. *By* J. HAMBLIN SMITH, M.A.

Crown 8vo. 6*s.*

A Key to Elementary Statics and Hydrostatics. *By* J. HAMBLIN SMITH, M.A.

Small 8vo. 3s. 6d.

Elements of Geometry. *By* J. HAMBLIN SMITH, M.A.
Containing Books 1 to 6, and portions of Books 11 and 12, of EUCLID, with Exercises and Notes, arranged with the Abbreviations admitted in the Cambridge University and Local Examinations.
Books 1 and 2, limp cloth, 1s. 6d., may be had separately.

Crown 8vo. 8s. 6d.

A Key to Elements of Geometry. *By* J. HAMBLIN SMITH, M.A.

Small 8vo. 1s.

Book of Enunciations for Hamblin Smith's Geometry, Algebra, Trigonometry, Statics, and Hydrostatics.

Small 8vo. 3s.

An Introduction to the Study of Heat. *By* J. HAMBLIN SMITH, M.A.

Crown 8vo. 5s.

Companion to Algebra. With numerous Examples. *By* L. MARSHALL, M.A., *Assistant Master at Charterhouse.*

CONTENTS.—Elementary Formulæ and Results—G. C. M., L. C. M., etc.—Evolution—Equations—Ratio, Proportion, and Variation—Indices, Surds, and Imaginary Quantities—Progressions—Permutations, Combinations, and Probabilities—Scales of Notation and Properties of Numbers—Binomial Theorem—Continued Fractions—Convergence and Divergence of Series—Indeterminate Co-efficients, and Partial Fractions—Summation of Series—Indeterminate Equations—General Theory of Equations—Miscellaneous Problems.

Crown 8vo. 6s.

The Principles of Dynamics. An Elementary Text-book for Science Students. *By* R. WORMELL, D.Sc., M.A., *Head Master of the City of London Middle-Class School.*

Small 8vo. 3s. 6d.

Army and Civil Service Examination Papers in Arithmetic, including Mensuration and Logarithms. With Arithmetical Rules, Tables, Formulæ and Answers, for the use of Students preparing for Examination. *By the* Rev. A. DAWSON CLARKE, M.A., *St. John's College, Cambridge.*

New Edition, Revised. Crown 8vo. 6s. 6d.

Arithmetic, Theoretical and Practical. *By* W. H. GIRDLESTONE, M.A., *of Christ's College, Cambridge.*
Also a School Edition. *Small 8vo. 3s. 6d.*

Latin Composition Books.

BENNETT'S First Latin Writer, *p.* 15.
—— First Latin Exercises, *p.* 15.
—— Second Latin Writer, *p.* 15.
RITCHIE'S First Steps in Latin, *p.* 16.
DAWE'S Beginner's Latin Exercise Book, *p.* 16.
ARNOLD'S Henry's First Latin Book, *p.* 18.
GEPP'S Arnold's Henry's First Latin Book, *p.* 18.
PROWDE SMITH'S Latin Prose Exercises, *p.* 19.
HAMBLIN SMITH'S Latin Exercises, *p.* 19.
ARNOLD'S Latin Prose, *p.* 18.
BRADLEY'S Arnold's Latin Prose, *p.* 18.
—— Aids to Latin Prose, *p.* 18.
SARGENT AND DALLIN'S Materials, *p.* 20.

Greek Composition Books.

SIDGWICK'S First Greek Writer, *p.* 23.
ARNOLD'S First Greek Book, *p.* 23.
MORICE'S Arnold's First Greek Book, *p.* 23.
RITCHIE AND MOORE'S Greek Method, *p.* 22.
ARNOLD'S Greek Prose, *p.* 24.
ABBOTT'S Arnold's Greek Prose, *p.* 24.
SIDGWICK'S Greek Prose, *p.* 23.
SARGENT AND DALLIN'S Materials, *p.* 25.

Latin Reading Books.

BENNETT'S Easy Latin Stories, *p.* 15.
—— Viri Illustres, *p.* 2.
—— Second Latin Reading Book, *p.* 15.
—— Selections from Vergil and Caesar, *p.* 15.
RITCHIE'S Preparatory Latin Reader, *p.* 2.
HEATLEY AND KINGDON'S Gradatim, *p.* 16.
—— Excerpta Facilia, *p.* 16.
TAYLOR'S Stories from Ovid, *p.* 19.

Greek Reading Books.

HEATLEY'S Græcula, *p.* 22.
MORICE'S Stories in Attic Greek, *p.* 26.
PHILLPOTT'S Stories from Herodotus, *p.* 25.
MOORE'S Selections from Thucydides, *p.* 2.
ABBOTT'S Selections from Lucian, *p.* 25.
MOBERLY'S Alexander the Great, *p.* 26.
SIDGWICK'S Scenes from Greek Plays, *p.* 26.

Latin and Greek Unseen Translation.

BENNETT'S Easy Latin Passages, *p.* 16.
TURNER'S Latin and Greek Passages, *p.* 2.
SARGENT'S Latin Passages, *p.* 20.
SPRATT AND PRETOR'S Latin and Greek Passages, *p.* 20.

LATIN

New Edition, Revised. Crown 8vo. 3s. 6d.

First Latin Writer. Comprising Accidence, the Easier Rules of Syntax illustrated by copious Examples, and progressive Exercises in Elementary Latin Prose, with Vocabularies. *By* G. L. BENNETT, M.A., *Head Master of Sutton Valence School.*

A KEY for the use of Tutors only. 5s.

Crown 8vo. 2s. 6d.

First Latin Exercises. Being the Exercises, with Syntax Rules and Vocabularies, from a "First Latin Writer." *By* G. L. BENNETT, M.A.

Crown 8vo. 1s. 6d.

Latin Accidence. From a "First Latin Writer." *By* G. L. BENNETT, M.A.

Second Edition. Crown 8vo. 3s. 6d.

Second Latin Writer. *By* G. L. BENNETT, M.A., *Head Master of Sutton Valence School.*

A KEY for the use of Tutors only. 5s.

New Edition, Revised. Crown 8vo. 2s. 6d.

Easy Latin Stories for Beginners. *By* G. L. BENNETT, M.A., *Head Master of Sutton Valence School.* With Vocabulary and Notes.

A KEY for the use of Tutors only. 5s.

Crown 8vo. 2s. 6d.

Second Latin Reading Book. Forming a continuation of "Easy Latin Stories for Beginners." *By* G. L. BENNETT, M.A., *Head Master of Sutton Valence School.*

A KEY for the use of Tutors only. 5s.

Small 8vo. 2s.

Selections from Caesar. The Gallic War. With Preface, Life of Caesar, Text, Notes, Geographical and Biographical Index, and Map of Gaul. *By* G. L. BENNETT, M.A., *Head Master of Sutton Valence School.*

Small 8vo. 1s. 6d.

Selections from the Aeneid of Vergil. With Introduction, Notes, &c. *By* G. L. BENNETT, M.A., *Head Master of Sutton Valence School.*

16mo. Paper covers, 1s.; or cloth, 1s. 4d.

Easy Graduated Latin Passages. For Translation into English, for use in Schools as Unseen Pieces. *By* G. L. BENNETT, M.A., *Head Master of Sutton Valence School.*

This book contains 256 pieces, the earlier portion being short sentences, the remainder consisting of graduated continuous pieces, including some of easy verse.

A KEY for the use of Tutors only. 3*s*. 6*d*.

Small 8vo. 1s. 6d.

Latin Grammar and Junior Scholarship Papers. *By* J. H. RAVEN, M.A., *Head Master of Fauconberge School, Beccles, Suffolk.*

A KEY for the use of Tutors only. 5*s*.

Small 8vo. 2s.

Easy Latin and Greek Grammar Papers. For the use of Public and Private Schools. *Prepared by* H. R. HEATLEY, M.A., *Assistant Master at Hillbrow School, Rugby.*

Second Edition. Crown 8vo. 1s. 6d.

First Steps in Latin. *By* F. RITCHIE, M.A., *The Beacon, Sevenoaks, and late Assistant Master in the High School, Plymouth.*

A KEY for the use of Tutors only. 3*s*. 6*d*.

New Edition. Small 8vo. 1s. 6d.

Gradatim. An Easy Translation Book for Beginners. With Vocabulary. *By* H. R. HEATLEY, M.A., *Assistant Master at Hillbrow School, Rugby, and* H. N. KINGDON, B.A., *Head Master of Dorchester Grammar School.*

A KEY for the use of Tutors only. 5*s*.

Second Edition. Small 8vo. 2s. 6d.

Excerpta Facilia. A Second Latin Translation Book, containing a Collection of Stories from various Latin Authors, with Notes at end, and a Vocabulary. *By* H. R. HEATLEY, M.A., *Assistant Master at Hillbrow School, Rugby, and* H. N. KINGDON, B.A., *Head Master of Dorchester Grammar School.*

Copies may also be had without the Vocabulary.

A KEY for the use of Tutors only. 5*s*.

Second Edition. Crown 8vo. 1s. 6d.

The Beginner's Latin Exercise Book. Affording Practice, oral and written, on Latin Accidence. With Vocabulary. *By* C. J. SHERWILL DAWE, B.A., *Lecturer and Assistant Chaplain at St. Mark's College, Chelsea.*

18mo.

Latin Texts. For use in schools, &c.

THE AENEID OF VERGIL. BOOKS I. II. III. IV. V. VII. VIII. IX. *2d.* each. BOOKS VI. X. XI. XII. *3d.* each.

THE GEORGICS OF VERGIL. BOOKS I.-IV. *2d.* each.

THE BUCOLICS OF VERGIL. *2d.*

Vergil. The Bucolics, Georgics, and Æneid in One Volume. *2s. 6d.*

CAESAR DE BELLO GALLICO. BOOKS I. V. VII. VIII. *3d.* each. BOOKS II. III. IV. VI. *2d.* each.

Caesar De Bello Gallico. In One Volume. *1s. 6d.*

Crown 8vo. On a card, 9d.

Elementary Rules of Latin Pronunciation. *By* ARTHUR HOLMES, M.A., *late Senior Fellow and Dean of Clare College, Cambridge.*

Sixth Edition. Crown 8vo. 3s. 6d.

Progressive Exercises in Latin Elegiac Verse. *By* C. G. GEPP, M.A., *Assistant Master at Bradfield College.*

A KEY for the use of Tutors only. *5s.*

Twelfth Edition. 12mo. 2s.

A First Verse Book. Being an Easy Introduction to the Mechanism of the Latin Hexameter and Pentameter. *By* THOMAS KERCHEVER ARNOLD, M.A.

A KEY for the use of Tutors only. *1s.*

Crown 8vo.

Clivus. Elementary Exercises in Latin Elegiac Verse. *By* A. C. AINGER, M.A., *Assistant Master at Eton College.*

PART I. *2s. 6d.* PART II. *2s. 6d.*

A KEY for the use of Tutors only. *3s. 6d.*

Crown 8vo. 3s. 6d.

An Elementary Latin Grammar. *By* J. HAMBLIN SMITH, M.A., *of Gonville and Caius College, and late Lecturer in Classics at St. Peter's College, Cambridge.*

Twenty-sixth Edition. 12mo. 3s.

Henry's First Latin Book. *By* T. K. ARNOLD, M.A.

A KEY for the use of Tutors only. 1s.

New Edition, Revised. 12mo. 3s.

Arnold's Henry's First Latin Book. *By* C.G.GEPP, M.A., *Assistant Master at Bradfield College, Author of* "Progressive Exercises in Latin Elegiac Verse."

A KEY for the use of Tutors only. 5s.

Twentieth Edition. 8vo. 6s. 6d.

A Practical Introduction to Latin Prose Composition. *By* THOMAS KERCHEVER ARNOLD, M.A.

A KEY for the use of Tutors only. 1s. 6d.

New Edition, Revised. Crown 8vo. 5s.

Arnold's Practical Introduction to Latin Prose Composition. *By* G. GRANVILLE BRADLEY, D.D., *Dean of Westminster, late Master of University College, Oxford, and formerly Master of Marlborough College.*

A KEY for the use of Tutors only. 5s.

Crown 8vo. 5s.

Aids to Writing Latin Prose. Containing 144 Exercises, with an Introduction comprising Preliminary Hints, Directions, Explanatory Matter, &c. *By* G. G. BRADLEY, D.D., *Dean of Westminster, and* T. L. PAPILLON, M.A., *Fellow and Tutor of New College, Oxford.*

A KEY for the use of Tutors only. 5s.

Crown 8vo.

The Aeneid of Vergil. *Edited, with Notes at the end, by* FRANCIS STORR, B.A., *Chief Master of Modern Subjects at Merchant Taylors' School.*

BOOKS I. and II. 2s. 6d. BOOKS XI. and XII. 2s. 6d.

Small 8vo. 1s. 6d.

Virgil, Georgics. BOOK IV. *Edited,* with Life, Notes, Vocabulary, and Index, *by* C. G. GEPP, M.A., *Assistant Master at Bradfield College.*

Third Edition, Revised. Crown 8vo. 3s. 6d.

Stories from Ovid in Elegiac Verse. With Notes and Marginal References to the "Public School Latin Primer." *By* R. W. TAYLOR, M.A., *Head Master of Kelly College, Tavistock.*

New Edition, Revised. Crown 8vo. 2s. 6d.

Stories from Ovid in Hexameter Verse. Metamorphoses. With Notes and Marginal References to the "Public School Latin Primer." *By* R. W. TAYLOR, M.A.

New Edition, Revised. 12mo. 2s. 6d.

Eclogæ Ovidianæ. From the Elegiac Poems. With English Notes. *By* THOMAS KERCHEVER ARNOLD, M.A.

Second Edition. Small 8vo. 2s.

Cicero de Amicitiâ. *Edited*, with Introduction and Notes, *by* ARTHUR SIDGWICK, M.A., *Fellow and Tutor of Corpus Christi College, Oxford.*

Small 8vo. 3s. 6d.

Cæsar. De Bello Gallico. BOOKS I.–III. *Edited by* J. H. MERRYWEATHER, M.A., *and* C. C. TANCOCK, M.A., *Assistant Masters at Charterhouse.*

BOOK I. separately. 2s.

Crown 8vo. 3s. 6d.

Exercises on the Elementary Principles of Latin Prose Composition. *By* J. HAMBLIN SMITH, M.A., *of Gonville and Caius College, and late Lecturer in Classics at St. Peter's College, Cambridge.* A KEY. 5s.

Small 8vo. 3s. 6d.

Livy. BOOK II. Chiefly from the text of MADVIG, with Notes, Translations, and Appendices. *Edited by* HENRY BELCHER, M.A., *Master of the Matriculation Class, King's College School, London.*

Fourth Edition. Crown 8vo. 2s. 6d.

Latin Prose Exercises. For Beginners, and Junior Forms of Schools. *By* R. PROWDE SMITH, B.A., *Assistant Master at Cheltenham College.*

Crown 8vo. 2s. 6d.

Latin Passages adapted for Practice in Unseen Translation. For the use of Middle Forms of Schools. *By* J. Y. SARGENT, M.A., *Fellow and Tutor of Hertford College, Oxford; and Editor of* "Materials and Models for Latin and Greek Prose Composition."

Crown 8vo.

Exercises in Translation at Sight. A Selection of Passages from Greek and Latin Authors. For the use of Students. *Arranged and translated by* A. W. SPRATT, M.A., *and* A. PRETOR, M.A., *Fellows of St. Catharine's College, Cambridge.*

Vol. I.—THE ORIGINAL PASSAGES. 4s. 6d.
Vol. II.—THE ENGLISH VERSIONS. [*Now ready.*

16mo.

A Latin-English Dictionary for Junior Forms of Schools. *By* C. G. GEPP, M.A., *Assistant Master at Bradfield College.* [*In the press.*

This work aims at supplying in a concise form and at a low price all the information required by boys in Middle Class Schools, or in the Junior Forms of Public Schools. Archaisms (with the exception of such as occur in the most commonly read authors), words peculiar to Plautus, and words found only in late or ecclesiastical Latin, have been accordingly excluded. On the other hand, Proper Names have been briefly yet adequately treated in alphabetical order in the body of the work. No effort has been spared to ensure completeness and accuracy, all references having been verified from the latest and most approved editions of modern scholars.

8vo. On a Card, 1s.

Outlines of Latin Sentence Construction. *By* E. D. MANSFIELD, M.A., *Assistant Master at Clifton College.*

Second Edition. Crown 8vo. 7s. 6d.

Classical Examination Papers. *Edited,* with Notes and References, *by* P. J. F. GANTILLON, M.A., *Classical Master at Cheltenham College.*

Or, interleaved with writing-paper, half-bound, 10s. 6d.

New Edition, re-arranged, with fresh Pieces and additional References.
Crown 8vo. 6s. 6d.

Materials and Models for Latin Prose Composition. *Selected and arranged by* J. Y. SARGENT, M.A., *Fellow and Tutor of Hertford College, Oxford, and* T. F. DALLIN, M.A., *late Tutor and Fellow of Queen's College, Oxford.*

A KEY to SELECTED PIECES (116), for the use of Tutors only. 5s.

Crown 8vo.

Terenti Comœdiæ. *Edited by* T. L. PAPILLON, M.A., *Fellow and Tutor of New College, Oxford.*

ANDRIA ET EUNUCHUS. With Introduction on Prosody. 4*s.* 6*d.*

Or separately, ANDRIA. 3*s.* 6*d.* EUNUCHUS. 3*s.*

Crown 8vo. 5*s.*

Juvenalis Satiræ. THIRTEEN SATIRES. *Edited by* G. A. SIMCOX, M.A., *Fellow of Queen's College, Oxford.*

Crown 8vo. 3*s.* 6*d.*

Persii Satiræ. *Edited by* A. PRETOR, M.A., *Fellow of St. Catharine's College, Cambridge.*

Crown 8vo. 7*s.* 6*d.*

Horati Opera. *By* J. M. MARSHALL, M.A., *Head Master of Durham School.*

VOL. I.—THE ODES, CARMEN SECULARE, AND EPODES.

Also separately, THE ODES. BOOKS I. to IV. 1*s.* 6*d.* each.

Crown 8vo.

Taciti Historiæ. *Edited by* W. H. SIMCOX, M.A., *Fellow of Queen's College, Oxford.*

BOOKS I. and II., 6*s.* BOOKS III., IV., and V., 6*s.*

Crown 8vo. Paper cover, 1*s. each.*

Plays of the Oratory School, Birmingham.

TERENCE—ANDRIA—PHORMIO—PINCERNA.

With English Notices to assist the representation.

GREEK

New Edition, Revised. Crown 8vo. 3s. 6d.

A Primer of Greek Grammar. With a Preface *by* JOHN PERCIVAL, M.A., LL.D., *President of Trinity College, Oxford; late Head Master of Clifton College.*

Or separately, crown 8vo. 2s. 6d.

Accidence. *By* EVELYN ABBOTT, M.A., LL.D., *Fellow and Tutor of Balliol College, Oxford; and* E. D. MANSFIELD, M.A., *Assistant Master at Clifton College.*

Crown 8vo. 1s. 6d.

Syntax. *By* E. D. MANSFIELD, M.A., *Assistant Master at Clifton College.*

New Edition. Crown 8vo. 3s. 6d.

A Practical Greek Method for Beginners. Being a Graduated application of Grammar to Translation and Composition. *By* F. RITCHIE, M.A., *The Beacon, Sevenoaks, and late Assistant Master at the High School, Plymouth; and* E. H. MOORE, M.A., *Assistant Master at the High School, Plymouth.*

A KEY for the use of Tutors only. 5s.

Crown 8vo. 2s. 6d.

A Manual of Greek Verbs. With Rules for the Formation of Tenses, and Tables of Verbs for Practice. *By* F. RITCHIE, M.A., *The Beacon, Sevenoaks, and late Assistant Master at the High School, Plymouth; and* E. H. MOORE, M.A., *Assistant Master at the High School, Plymouth.*

Small 8vo. 1s. 6d.

Graecula. A First Book of Greek Translation. Rules, Short Sentences, Stories for Translation, and a Vocabulary. *By* H. R. HEATLEY, M.A., *Assistant Master at Hillbrow School, Rugby.*

A KEY for the use of Tutors only. 5s.

Small 8vo. 2s.

Easy Latin and Greek Grammar Papers. For the use of Public and Private Schools. *Prepared by* H. R. HEATLEY, M.A., *Assistant Master at Hillbrow School, Rugby.*

New Edition, Revised. Crown 8vo. 3s. 6d.

A First Greek Writer. *By* ARTHUR SIDGWICK, M.A., *Fellow and Tutor of Corpus Christi College, Oxford; late Assistant Master at Rugby School, and Fellow of Trinity College, Cambridge.*

A KEY for the use of Tutors only. 5*s.*

Fourth Edition, Revised. Crown 8vo. 5s.

An Introduction to Greek Prose Composition. With Exercises. *By* ARTHUR SIDGWICK, M.A.

A KEY for the use of Tutors only. 5*s.*

Second Edition, Revised. Crown 8vo. 5s.

An Introduction to Greek Verse Composition. With Exercises. *By* ARTHUR SIDGWICK, M.A., *Fellow and Tutor of Corpus Christi College, Oxford; and* F. D. MORICE, M.A., *Assistant Master at Rugby School, and Fellow of Queen's College, Oxford.*

A KEY for the use of Tutors only. 5*s.*

Sixth Edition. 12mo. 5s.

The First Greek Book. On the plan of *Henry's First Latin Book. By* THOMAS KERCHEVER ARNOLD, M.A.

A KEY for the use of Tutors only. 1*s.* 6*d.*

New Edition, Revised. Crown 8vo. 3s. 6d.

Arnold's First Greek Book. *By* FRANCIS DAVID MORICE, M.A., *Assistant Master at Rugby School, and Fellow of Queen's College, Oxford.*

A KEY for the use of Tutors only. 5*s.*

Cloth limp, 8vo. 6d.

Elementary Card on Greek Prepositions. *By* Rev. E. PRIESTLAND, M.A., *Spondon House School, Derbyshire.*

Crown 8vo. 9d.

A Short Greek Syntax. Extracted from "Xenophon's Anabasis, with Notes." *By* R. W. TAYLOR, M.A., *Head Master of Kelly College, Tavistock.*

Crown 8vo. 8s. 6d.

A Syntax of Attic Greek for the use of Students and Schools. *By* F. E. THOMPSON, M.A., *Assistant Master at Marlborough College.*

Third Edition. Imperial 16mo. 8s. 6d.

Mdduig's Syntax of the Greek Language, especially of the Attic Dialect. For the use of Schools. *Edited by* THOMAS KERCHEVER ARNOLD, M.A.

Tenth Edition. 8vo. 5s. 6d.

A Practical Introduction to Greek Accidence. *By* THOMAS KERCHEVER ARNOLD, M.A.

Fourteenth Edition. 8vo. 5s. 6d.

A Practical Introduction to Greek Prose Composition. *By* THOMAS KERCHEVER ARNOLD, M.A.

A KEY for the use of Tutors only. 1s. 6d.

New Edition, Revised. Crown 8vo. 3s. 6d.

Arnold's Practical Introduction to Greek Prose Composition. *By* EVELYN ABBOTT, M.A., LL.D., *Fellow and Tutor of Balliol College, Oxford.*

A KEY for the use of Tutors only. 3s. 6d.

Crown 8vo. 4s. 6d.

Elements of Greek Accidence. *By* EVELYN ABBOTT, M.A., LL.D., *Fellow and Tutor of Balliol College, Oxford.*

Crown 8vo. 4s. 6d.

An Elementary Greek Grammar. *By* J. HAMBLIN SMITH, M.A., *of Gonville and Caius College, and late Lecturer in Classics at St. Peter's College, Cambridge.*

Cloth limp, 8vo. 1s.

A Table of Irregular Greek Verbs, classified according to the arrangement of Crusius's Greek Grammar. *By* FRANCIS STORR, B.A., *Chief Master of Modern Subjects at Merchant Taylors' School, and late Assistant Master at Marlborough College.*

Crown 8vo.

Exercises in Translation at Sight. A Selection of Passages from Greek and Latin Authors. For the use of Students. *Arranged and Translated by* A. W. SPRATT, M.A., *and* A. PRETOR, M.A., *Fellows of St. Catharine's College, Cambridge.*

Vol. I.—THE ORIGINAL PASSAGES. 4*s.* 6*d.*
Vol. II.—THE ENGLISH VERSIONS.

Crown 8vo. 7*s.* 6*d.*

Etyma Graeca. An Etymological Lexicon of Classical Greek. *By* E. R. WHARTON, M.A., *Lecturer and late Fellow of Jesus College, Oxford.*

Second Edition. Crown 8vo. 7*s.* 6*d.*

Classical Examination Papers. *Edited,* with Notes and References, *by* P. J. F. GANTILLON, M.A., *Classical Master at Cheltenham College.*

Or interleaved with writing-paper, half-bound, 10*s.* 6*d.*

Second Edition, containing fresh Pieces and additional References.
Crown 8vo. 5*s.*

Materials and Models for Greek Prose Composition. *Selected and arranged by* J. Y. SARGENT, M.A., *Fellow and Tutor of Hertford College, Oxford; and* T. F. DALLIN, M.A.

A KEY to SELECTED PIECES (92), for the use of Tutors only. 7*s.* 6*d.*

Crown 8vo. 2*s.*

Iophon: An Introduction to the Art of Writing Greek Iambic Verses. *By the* WRITER *of* "Nuces" *and* "Lucretilis."

Fifth Edition. Crown 8vo. 1*s.* 6*d.*

Stories from Herodotus. The Tales of Rhampsinitus and Polycrates, and the Battle of Marathon and the Alcmaeonidae. In Attic Greek. *Edited by* J. SURTEES PHILLPOTTS, M.A., *Head Master of Bedford Grammar School.*

New Edition, Revised. Small 8vo. 3*s.* 6*d.*

Selections from Lucian. With English Notes. *By* EVELYN ABBOTT, M.A., LL.D., *Fellow and Tutor of Balliol College, Oxford.*

Small 8vo. 1s. 6d. each.

Scenes from Greek Plays. RUGBY EDITION.

Abridged and adapted for the use of Schools, by ARTHUR SIDGWICK, M.A., *Fellow of Corpus Christi College, Oxford, late Assistant Master at Rugby School, and Fellow of Trinity College, Cambridge.*

Aristophanes.

THE CLOUDS. THE FROGS. THE KNIGHTS. PLUTUS.

Euripides.

IPHIGENIA IN TAURIS. THE CYCLOPS. ION.
ELECTRA. ALCESTIS. BACCHÆ. HECUBA.

Fourth Edition. Crown 8vo. 3s. 6d.

Stories in Attic Greek.

Forming a Greek Reading Book for the use of Junior Forms in Schools. With Notes and Vocabulary. *By* FRANCIS DAVID MORICE, M.A., *Assistant Master at Rugby School, and Fellow of Queen's College, Oxford.*

New Edition. Crown 8vo.

The Anabasis of Xenophon.

Edited, with Preface, Introduction, Historical Sketch, Itinerary, Syntax Rules, Notes, Indices, Vocabularies, and Maps, *by* R. W. TAYLOR, M.A., *Head Master of Kelly College, Tavistock, and late Fellow of St. John's College, Cambridge.*

BOOKS I. and II. 3s. 6d. BOOKS III. and IV. 3s. 6d.
Also separately, BOOK I., 2s. 6d.; BOOK II., 2s.

Crown 8vo. 2s. 6d.

Xenophon's Agesilaus.

Edited, with Syntax Rules, and References, Notes, and Indices, *by* R. W. TAYLOR, M.A.

Second Edition. Small 8vo. 2s.

Xenophon's Memorabilia.

BOOK I., with a few omissions. *Edited*, with an Introduction and Notes, *by* C. E. MOBERLY, M.A., *formerly Scholar of Balliol College, Oxford.*

New Edition. Small 8vo. 2s.

Alexander the Great in the Punjaub.

Adapted from Arrian, Book V. An Easy Greek Reading Book. *Edited*, with Notes and a Map, *by* C. E. MOBERLY, M.A., *formerly Scholar of Balliol College, Oxford.*

Small 8vo.

Homer's Iliad. *Edited,* with Notes at the end for the Use of Junior Students, *by* ARTHUR SIDGWICK, M.A., *Fellow of Corpus Christi College, Oxford; late Assistant Master at Rugby School, and Fellow of Trinity College, Cambridge.*

BOOKS I. and II. 2*s.* 6*d.*

BOOK XXI. 1*s.* 6*d.* BOOK XXII. 1*s.* 6*d.*

Small 8vo. 2*s.*

Homer without a Lexicon, for Beginners. ILIAD, BOOK VI. *Edited,* with Notes giving the meanings of all the less common words, *by* J. SURTEES PHILLPOTTS, M.A., *Head Master of Bedford Grammar School.*

Fifth Edition. 12*mo.* 3*s.* 6*d.*

Homer for Beginners. ILIAD, BOOKS I.–III. With English Notes. *By* THOMAS KERCHEVER ARNOLD, M.A.

Fifth Edition. 12*mo.* 12*s.*

The Iliad of Homer. With English Notes and Grammatical References. *By* THOMAS KERCHEVER ARNOLD, M.A.

Crown 8vo. 6*s.*

The Iliad of Homer. BOOKS I.–XII. From the Text of Dindorf. With Preface and Notes. *By* S. H. REYNOLDS, M.A., *late Fellow and Tutor of Brasenose College, Oxford.*

8vo. 18*s.*

Myths of the Odyssey in Art and Literature. Illustrated with Outline Drawings. *By* J. E. HARRISON.

New Edition. 12*mo.* 9*s.*

A Complete Greek and English Lexicon for the Poems of Homer and the Homeridæ. *By* G. CH. CRUSIUS. Translated from the German. *Edited by* T. K. ARNOLD, M.A.

Crown 8vo. 4*s.* 6*d.*

Isocratis Orationes. AD DEMONICUM ET PANEGYRICUS. *Edited by* JOHN EDWIN SANDYS, M.A., *Fellow and Tutor of St. John's College, Cambridge, and Public Orator of the University.*

8vo. 16s.

Hellenica. A Collection of Essays on Greek Poetry, Philosophy, History, and Religion. *Edited by* EVELYN ABBOTT, M.A., LL.D., *Fellow and Tutor of Balliol College, Oxford.*

CONTENTS.—Aeschylus. E. Myers, M.A.—The Theology and Ethics of Sophocles. E. Abbott, M.A., LL.D.—System of Education in Plato's Republic. R. L. Nettleship, M.A.—Aristotle's Conception of the State. A. C. Bradley, M.A.—Epicurus. W. L. Courtney, M.A.—The Speeches of Thucydides. R. C. Jebb, M.A., LL.D.—Xenophon. H. G. Dakyns, M.A.—Polybius. J. L. S. Davidson, M.A.—Greek Oracles. F. W. H. Myers, M.A.

8vo. 18s.

The Antiquities of Greece. THE STATE. Translated from the German of G. F. SCHOEMANN. *By* E. G. HARDY, M.A., *Head Master of the Grammar School, Grantham; and* J. S. MANN, M.A., *Fellow of Trinity College, Oxford.*

Crown 8vo.

Herodoti Historia. *Edited by* H. G. WOODS, M.A., *Fellow of Trinity College, Oxford.*

BOOK I. 6s. BOOK II. 5s.

12mo.

Demosthenes. *Edited*, with English Notes and Grammatical References, *by* THOMAS KERCHEVER ARNOLD, M.A.

OLYNTHIAC ORATIONS. *Third Edition.* 3s.
ORATION ON THE CROWN. *Second Edition.* 4s. 6d.

Crown 8vo. 5s.

Demosthenis Orationes Privatæ. DE CORONA. *Edited by* ARTHUR HOLMES, M.A., *late Senior Fellow and Dean of Clare College, Cambridge.*

Crown 8vo.

Demosthenis Orationes Publicæ. *Edited by* G. H. HESLOP, M.A., *late Fellow and Assistant Tutor of Queen's College, Oxford; Head Master of St. Bees.*

OLYNTHIACS, 2s. 6d. } or, in One Volume, 4s. 6d.
PHILIPPICS, 3s. }
DE FALSA LEGATIONE, 6s.

Crown 8vo.

Aristophanis Comœdiæ. *Edited by* W. C. GREEN, M.A., *late Fellow of King's College, Cambridge; Assistant Master at Rugby School.*

THE ACHARNIANS and THE KNIGHTS. 4s.
THE CLOUDS. 3s. 6d. THE WASPS. 3s. 6d.

Second Edition, Revised and Enlarged. Crown 8vo. 10*s.* 6*d.*

An Introduction to Aristotle's Ethics. BOOKS I.–IV. (BOOK X., c. vi.–ix. in an Appendix). With a Continuous Analysis and Notes. Intended for the use of Beginners and Junior Students. *By the* Rev. EDWARD MOORE, B.D., *Principal of St. Edmund Hall, and late Fellow and Tutor of Queen's College, Oxford.*

Second Edition. Crown 8vo. 3*s.* 6*d.*

Selections from Aristotle's Organon. *Edited by* JOHN R. MAGRATH, D.D., *Provost of Queen's College, Oxford.*

12*mo.*

Sophocles. *Edited by* T. K. ARNOLD, M.A., ARCHDEACON PAUL, *and* HENRY BROWN, M.A.

AJAX. 3*s.* OEDIPUS TYRANNUS. 4*s.*

Crown 8vo.

Sophoclis Tragœdiæ. *Edited by* R. C. JEBB, M.A., LL.D., *Professor of Greek at the University of Glasgow, late Fellow and Tutor of Trinity College, Cambridge.*

ELECTRA. 3*s.* 6*d.* AJAX. 3*s.* 6*d.*

Crown 8vo. 8*s.* 6*d.*

Sophocles. Translated into English Verse. *By* ROBERT WHITELAW, M.A., *Assistant Master in Rugby School; late Fellow of Trinity College, Cambridge.*

Crown 8vo. 6*s.*

Thucydidis Historia. BOOKS I. and II. *Edited by* CHARLES BIGG, D.D., *late Senior Student and Tutor of Christ Church, Oxford; formerly Principal of Brighton College.*

Crown 8vo. 6*s.*

Thucydidis Historia. BOOKS III. and IV. *Edited by* G. A. SIMCOX, M.A., *Fellow of Queen's College, Oxford.*

Sixth Edition. 8vo. 21*s.*

A Copious Phraseological English-Greek Lexicon. *Founded on a work prepared by* J. W. FRÄDERSDORFF, Ph.D., *late Professor of Modern Languages, Queen's College, Belfast. Revised, Enlarged, and Improved by* THOMAS KERCHEVER ARNOLD, M.A., *and* HENRY BROWNE, M.A.

Third Edition. Crown 8vo. 2s. 6d.

Short Notes on the Greek Text of the Gospel of St. Mark. *By* J. HAMBLIN SMITH, M.A., *of Gonville and Caius College, Cambridge.*

Crown 8vo. 4s. 6d.

Notes on the Greek Text of the Acts of the Apostles. *By* J. HAMBLIN SMITH, M.A., *of Gonville and Caius College, Cambridge.*

Crown 8vo. 6s.

Notes on the Gospel According to S. Luke. *By the* REV. ARTHUR CARR, M.A., *Assistant Master at Wellington College, late Fellow of Oriel College, Oxford.*

New Edition. 4 vols. 8vo. 102s.

The Greek Testament. With a Critically Revised Text; a Digest of Various Readings; Marginal References to Verbal and Idiomatic Usage; Prolegomena; and a Critical and Exegetical Commentary. For the use of Theological Students and Ministers. *By* HENRY ALFORD, D.D., *late Dean of Canterbury.*

The Volumes are sold separately, as follows:—

Vol. I.—THE FOUR GOSPELS. 28s.
Vol. II.—ACTS TO 2 CORINTHIANS. 24s.
Vol. III.—GALATIANS TO PHILEMON. 18s.
Vol. IV.—HEBREWS TO REVELATION. 32s.

New Edition. 2 vols. Imperial 8vo. 60s.

The Greek Testament. With Notes, Introductions, and Index. *By* CHR. WORDSWORTH, D.D., *Bishop of Lincoln.*

The Parts may be had separately, as follows:

THE GOSPELS. 16s.
THE ACTS. 8s.
ST. PAUL'S EPISTLES. 23s.
GENERAL EPISTLES, REVELATION, AND INDEX. 16s.

CATENA CLASSICORUM

Crown 8vo.

Aristophanis Comœdiae. *By* W. C. GREEN, M.A.

THE ACHARNIANS AND THE KNIGHTS. 4*s.*

THE WASPS. 3*s.* 6*d.* THE CLOUDS. 3*s.* 6*d.*

Demosthenis Orationes Publicae. *By* G. H. HESLOP, M.A.

THE OLYNTHIACS. 2*s.* 6*d.*
THE PHILIPPICS. 3*s.* } or, in One Volume, 4*s.* 6*d.*

DE FALSA LEGATIONE. 6*s.*

Demosthenis Orationes Privatae. *By* A. HOLMES, M.A.

DE CORONA. 5*s.*

Herodoti Historia. *By* H. G. WOODS, M.A.

BOOK I., 6*s.* BOOK II., 5*s.*

Homeri Ilias. *By* S. H. REYNOLDS, M.A.

BOOKS I.–XII. 6*s.*

Horati Opera. *By* J. M. MARSHALL, M.A.

THE ODES, CARMEN SECULARE, AND EPODES. 7*s.* 6*d.*

THE ODES. BOOKS I. to IV. separately. 1*s.* 6*d.* each.

Isocratis Orationes. *By* JOHN EDWIN SANDYS, M.A.

AD DEMONICUM ET PANEGYRICUS. 4*s.* 6*d.*

Juvenalis Satirae. *By* G. A. SIMCOX, M.A. 5*s.*

Persii Satirae. *By* A. PRETOR, M.A. 3*s.* 6*d.*

Sophoclis Tragoediae. *By* R. C. JEBB, M.A.

THE ELECTRA. 3*s.* 6*d.* THE AJAX. 3*s.* 6*d.*

Taciti Historiae. *By* W. H. SIMCOX, M.A.

BOOKS I. and II., 6*s.* BOOKS III. IV. and V., 6*s.*

Terenti Comœdiæ. *By* T. L. PAPILLON, M.A.

ANDRIA AND EUNUCHUS. With Introduction on Prosody. 4*s.* 6*d.* *Or separately.*

ANDRIA. With Introduction on Prosody. 3*s.* 6*d.*

EUNUCHUS. 3*s.*

Thucydidis Historia.

BOOKS I. and II. *By* CHARLES BIGG, D.D. 6*s.*

BOOKS III. and IV. *By* G. A. SIMCOX, M.A. 6*s.*

DIVINITY

New and Revised Editions. Small 8vo. 3s. 6d. each. Sold separately.

Manuals of Religious Instruction. *Edited by*
JOHN PILKINGTON NORRIS, D.D., *Archdeacon of Bristol.*

The Old Testament. | The New Testament.
The Prayer Book.

Cheap Edition. Small 8vo. 1s. 6d. each.

Keys to Christian Knowledge. *By the* Rev. J. H.
BLUNT, D.D., *Editor of the* "Annotated Book of Common Prayer."

The Holy Bible.	**The Church Catechism.**
The Book of Common Prayer.	**Church History, Ancient.**
	Church History, Modern.

By JOHN PILKINGTON NORRIS, D.D., *Archdeacon of Bristol.*

The Four Gospels.	**The Acts of the Apostles.**

18mo. 1s. 6d.

Easy Lessons Addressed to Candidates for Confirmation. *By* J. P. NORRIS, D.D., *Archdeacon of Bristol.*

New Edition. Small 8vo. 1s. 6d.

A Manual of Confirmation. *By* EDWARD MEYRICK GOULBURN, D.D., *Dean of Norwich.*

Crown 8vo. 7s. 6d.

Some Helps for School Life. Sermons preached at Clifton College, 1862–1879. *By* J. PERCIVAL, M.A., LL.D., *President of Trinity College, Oxford, and late Head Master of Clifton College.*

Crown 8vo. 5s.

Modern Laodiceans and other Sermons. Chiefly preached to Bradfield Boys. *By the* Rev. H. B. GRAY, M.A., *Warden of Bradfield College, Berks.*

Crown 8vo. 1s. Cloth limp, 1s. 6d.

Study of the Church Catechism. Adapted for use as a Class Book. *By* C. J. SHERWILL DAWE, M.A., *Lecturer and Assistant Chaplain at St. Mark's College, Chelsea.*

New Edition. Small 8vo. 3s. 6d.

Household Theology. A Handbook of Religious Information respecting the Holy Bible, the Prayer Book, the Church, the Ministry, Divine Worship, the Creeds, &c. &c. *By the* Rev. JOHN HENRY BLUNT, D.D., F.S.A.

Second Edition, Revised. Crown 8vo. 7s. 6d.

Rudiments of Theology. A First Book for Students. *By* JOHN PILKINGTON NORRIS, D.D., *Archdeacon of Bristol.*

16mo, 1s. 6d.; Paper Covers, 1s.; or in Three Parts, 6d. each.

The Young Churchman's Companion to the Prayer Book. *By the* Rev. J. W. GEDGE, M.A., *Diocesan Inspector of Schools for the Archdeaconry of Surrey.*

PART I.—MORNING AND EVENING PRAYER, AND LITANY.
PART II.—BAPTISMAL AND CONFIRMATION SERVICES.
PART III.—THE HOLY COMMUNION.

Second Edition. Large type. 24mo. 1s.

Prayers for a Young Schoolboy. *By the* Rev. E. B. PUSEY, D.D. *Edited, with a Preface, by* H. P. LIDDON, D.D., *Canon Residentiary of St. Paul's.*

Second Edition. 18mo. 1s. 6d.

The Way of Life. A Book of Prayers and Instruction for the Young at School. With a Preparation for Holy Communion. *Compiled by a Priest. Edited by the* Rev. T. T. CARTER, M.A.

16mo. 2s. 6d.

A Plain Exposition of the Thirty-nine Articles of the Church of England. For the use of Schools. *By the* Rev. WILLIAM BAKER, D.D., *Head Master of Merchant Taylors' School, and Prebendary of St. Paul's.*

Crown 16mo. Cloth limp. 1s. 6d.

A Manual of Devotion, chiefly for the Use of Schoolboys. *By* WILLIAM BAKER, D.D., *Head Master of Merchant Taylors' School. With Preface by* J. R. WOODFORD, D.D., *Lord Bishop of Ely.*

FRENCH

Crown 8vo.

Exercises in French Syntax. With Rules. *By* G. SHARP, M.A., *Assistant Master at Marlborough College.*

Crown 8vo.

Passages for Unseen French Translation. *By* C. H. PARRY, M.A., *Assistant Master at Charterhouse.*

New Edition. Small 8vo. 2*s.*

A Graduated French Reader. With an Introduction on the Pronunciation of Consonants and the Connection of Final Letters, a Vocabulary, and Notes, and a Table of Irregular Verbs with the Latin Infinitives. *By* PAUL BARBIER, *Professor at University College, Cardiff.*

Crown 8vo.

The Campaigns of Napoleon. *The Text (in French) from* M. THIERS' "Histoire de la Révolution Française," *and* "Histoire du Consulat et de l'Empire." *Edited*, with English Notes and Maps, for the use of Schools, *by* EDWARD E. BOWEN, M.A., *Master of the Modern Side, Harrow School.*

ARCOLA. 4*s.* 6*d.* MARENGO. 4*s.* 6*d.*
JENA. 3*s.* 6*d.* WATERLOO. 6*s.*

New Editions. Crown 8vo. 3*s.* 6*d. each.*

Selections from Modern French Authors. *Edited*, with English Notes and Introductory Notice, *by* HENRI VAN LAUN, *Translator of* TAINE'S "History of English Literature."

HONORÉ DE BALZAC. H. A. TAINE.

Small 8vo. 2*s.*

La Fontaine's Fables. BOOKS I. and II. *Edited*, with English Notes at the end, *by* Rev. P. BOWDEN-SMITH, M.A., *Assistant Master at Rugby School.*

Sixth Edition. 12*mo.* 5*s.* 6*d.*

The First French Book. *By* T. K. ARNOLD, M.A.

KEY, 2*s.* 6*d.*

MISCELLANEOUS

With Maps. Small 8vo.

A Geography, Physical, Political, and Descriptive. For Beginners. *By* L. B. LANG. *Edited by the* Rev. M. CREIGHTON, M.A., LL.D., *late Fellow and Tutor of Merton College, Oxford.*

VOL. I. THE BRITISH EMPIRE. 2*s.* 6*d.*

PART I. THE BRITISH ISLES. 1s. 6d. PART II. THE BRITISH POSSESSIONS. 1s. 6d.

VOL. II. THE CONTINENT OF EUROPE. 3*s.*

VOL. III. ASIA, AFRICA, AND AMERICA. 2*s.*

Small 8vo. 2s. 6d. each part.

Modern Geography for the Use of Schools. *By* C. E. MOBERLY, M.A., *formerly Scholar of Balliol College, Oxford.*

PART I. NORTHERN EUROPE.

PART II. THE MEDITERRANEAN & ITS PENINSULAS.

Crown 8vo. 3s. 6d.

At Home and Abroad; or, First Lessons in Geography. *By* J. K. LAUGHTON, M.A., F.R.A.S., F.R.G.S., *Mathematical Instructor and Lecturer at the Royal Naval College.*

Second Edition. Crown 8vo. 2s. 6d.

The Chorister's Guide. *By* W. A. BARRETT, Mus. Bac. Oxon., *Vicar Choral of St. Paul's Cathedral, Author of* "Flowers and Festivals," *&c.*

Crown 8vo. 2s. 6d.

An Introduction to Form and Instrumentation. For the use of Beginners in Composition. *By* W. A. BARRETT, Mus. Bac. Oxon., *Vicar Choral of St. Paul's Cathedral.*

Seventh Edition, Revised. 12mo. 7s. 6d.

The First Hebrew Book. *By* T. K. ARNOLD, M.A.

KEY, 3*s.* 6*d.*

By J. Hamblin Smith.

Elementary Algebra. *Small 8vo,* 3*s.* *Without Answers,* 2*s.* 6*d.*

Key to Elementary Algebra. *Crown 8vo,* 9*s.*

Exercises on Algebra. *Small 8vo,* 2*s.* 6*d.*

Arithmetic. *Small 8vo,* 3*s.* 6*d.*

Key to Arithmetic. *Crown 8vo,* 9*s.*

Elements of Geometry. *Small 8vo,* 3*s.* 6*d.*
Books I. and II., limp cloth, price 1*s.* 6*d.*, may be had separately.

Key to Elements of Geometry. *Crown 8vo,* 8*s.* 6*d.*

Trigonometry. *Small 8vo,* 4*s.* 6*d.*

Key to Trigonometry. *Crown 8vo,* 7*s.* 6*d.*

Elementary Statics. *Small 8vo,* 3*s.*

Elementary Hydrostatics. *Small 8vo,* 3*s.*

Key to Elementary Statics and Hydrostatics. *Crown 8vo,* 6*s.*

Book of Enunciations for HAMBLIN SMITH'S Geometry, Algebra, Trigonometry, Statics, and Hydrostatics. *Small 8vo,* 1*s.*

An Introduction to the Study of Heat. *Small 8vo,* 3*s.*

Latin Grammar. *Crown 8vo,* 3*s.* 6*d.*

Exercises on the Elementary Principles of Latin Prose Composition. *Crown 8vo,* 3*s.* 6*d.*

Key to Exercises on Latin Prose Composition. *Crown 8vo,* 5*s.*

An Elementary Greek Grammar. *Crown 8vo,* 4*s.* 6*d.*

The Rudiments of English Grammar and Composition. *Crown 8vo,* 2*s.* 6*d.*

Notes on the Greek Text of the Acts of the Apostles. *Crown 8vo,* 4*s.* 6*d.*

Notes on the Greek Text of the Gospel of St. Mark. *Crown 8vo,* 2*s.* 6*d.*

By Arthur Sidgwick.

An Introduction to Greek Prose Composition. *Crown 8vo,* 5*s.* A KEY, 5*s.*

An Introduction to Greek Verse Composition. *Crown 8vo,* 5*s.* A KEY, 5*s.*

A First Greek Writer. *Crown 8vo,* 3*s.* 6*d.* A KEY, 5*s.*

Cicero de Amicitia. *Small 8vo,* 2*s.*

Homer's Iliad. *Small 8vo.* BOOKS I. and II., 2*s.* 6*d.* BOOK XXI., 1*s.* 6*d.* BOOK XXII., 1*s.* 6*d.*

Scenes from Greek Plays. *Small 8vo, each* 1*s.* 6*d.*

ARISTOPHANES: The Clouds, The Frogs, The Knights, Plutus.

EURIPIDES: Iphigenia in Tauris, The Cyclops, Ion, Electra, Alcestis, Bacchæ, Hecuba.

By George L. Bennett.

First Latin Writer. *Crown 8vo,* 3*s.* 6*d.* A KEY, 5*s.*

First Latin Exercises. *Crown 8vo,* 2*s.* 6*d.*

First Latin Accidence. *Crown 8vo,* 1*s.* 6*d.*

Second Latin Writer. *Crown 8vo,* 3*s.* 6*d.* A KEY, 5*s.*

Easy Latin Stories for Beginners. *Cr. 8vo,* 2*s.* 6*d.* A KEY, 5*s.*

Viri Illustres Urbis Romae. *Small 8vo.*

Second Latin Reading Book. *Cr. 8vo,* 2*s.* 6*d.* A KEY, 5*s.*

Selections from Caesar. The Gallic War. *Small 8vo,* 2*s.*

Selections from Vergil. *Small 8vo,* 1*s.* 6*d.*

Easy Unseen Latin Passages. 16*mo,* 1*s.* A KEY, 3*s.* 6*d.*

By R. W. Taylor.

Xenophon's Anabasis. *Crown 8vo.* Books I. and II., 3*s.* 6*d.*; III. and IV., 3*s* 6*d.* Also separately, Book I., 2*s.* 6*d.* II., 2*s.*

Xenophon's Agesilaus. *Crown 8vo,* 2*s.* 6*d.*

A Short Greek Syntax. *Crown 8vo,* 9*d.*

Stories from Ovid in Elegiac Verse. *Crown 8vo,* 3*s.* 6*d.*

Stories from Ovid in Hexameter Verse. METAMORPHOSES. *Crown 8vo,* 2*s.* 6*d.*

Scott's Lady of the Lake. Forming a Volume of the "English School Classics." *Small 8vo,* 2*s.*; *or in Three Parts, each* 9*d.*

INDEX

INDEX.

RIVINGTON'S EDUCATIONAL LIST

Arnold's Henry's First Latin Book. By C. G. GEPP. 3s.
[The original Edition is still on sale.]

Arnold's Latin Prose Composition. By G. G. BRADLEY. 5s.
[The original Edition is still on sale.]

First Latin Writer. By G. L. BENNETT. 3s. 6d.

Or separately—

First Latin Exercises. 1s. 6d.

Latin Accidence. 1s. 6d.

Second Latin Writer. By G. L. BENNETT. 3s. 6d.

Easy Latin Stories for Beginners. By G. L. BENNETT. 2s. 6d.

Second Latin Reading Book. By G. L. BENNETT. 2s. 6d.

Selections from Cæsar. By G. L. BENNETT. 2s.

Selections from Vergil. By G. L. BENNETT. 1s. 6d.

Cæsar de Bello Gallico. Books I.—III. By J. MERRYWEATHER and C. TANCOCK. 3s. 6d. Book I. separately, 2s.

Gradatim. An Easy Latin Translation Book. By H. R. HEATLEY and H. N. KINGDON. 1s. 6d.

Excerpta Facilia. A Second Latin Translation Book. By H. R. HEATLEY and H. N. KINGDON. 2s. 6d.

First Steps in Latin. By F. RITCHIE. 1s. 6d.

Arnold's Greek Prose Composition. By E. ABBOTT. 3s. 6d.
[The original Edition is still on sale.]

A Primer of Greek Grammar. By E. ABBOTT and E. D. MANSFIELD. 3s. 6d.

Separately—

Syntax. 1s. 6d. *Accidence.* 2s. 6d.

A Practical Greek Method for Beginners. THE SIMPLE SENTENCE. By F. RITCHIE and E. H. MOORE. 3s. 6d.

Stories in Attic Greek. By F. D. MORICE. 3s. 6d.

A First Greek Writer. By A. SIDGWICK. 3s. 6d.

An Introduction to Greek Prose Composition. By A. SIDGWICK. 5s.

An Introduction to Greek Verse Composition. By A. SIDGWICK and F. D. MORICE. 5s.

Homer's Iliad. By A. SIDGWICK. Books I. and II., 2s. 6d.—Book XXI., 1s. 6d.—Book XXII., 1s. 6d.

The Anabasis of Xenophon. By R. W. TAYLOR. Books I. and II., 3s. 6d. Separately, Book I., 2s. 6d.; Book II., 2s.—Books III. and IV., 3s. 6d.

Xenophon's Agesilaus. By R. W. TAYLOR. 2s. 6d.

Stories from Ovid in Elegiac Verse. By R. W. TAYLOR. 3s. 6d.

Stories from Ovid in Hexameter Verse. By R. W. TAYLOR. 2s. 6d.

Waterloo Place, Pall Mall, London.